

UNIVERSITY OF CALIFORNIA, BERKELEY

BERKELEY • DAVIS • IRVINE • LOS ANGELES • RIVERSIDE • SAN DIEGO • SAN FRANCISCO



SANTA BARBARA • SANTA CRUZ

DEPARTMENT OF MATHEMATICS

BERKELEY, CALIFORNIA 94720

26 March 1981

Dear Sir,

Enclosed, please find 3 copies of our essay entitled "Non-Self-Dual Nonlinear Gravities" which is to be entered in your 1981 Gravity Essay Contest. Thank-you for your consideration.

Sincerely
James A. Eisenberg
Jim Eisenberg +
Philip B. Yasskin

1111111111 1111111111 1111111111 1111111111



THE UNIVERSITY OF WISCONSIN—MILWAUKEE/P.O. Box 413, Milwaukee, Wisconsin 53201

DEPARTMENT OF PHYSICS
COLLEGE OF LETTERS AND SCIENCE
AC (414) 963-4474

Gravity Research Foundation
58 Middle Street
Gloucester, Massachusetts

Dear Sirs:

Enclosed are three copies of a manuscript entitled "Half-integral Spin from Quantum Gravity", by myself and R.D. Sorkin, which is intended for this year's essay competition. Thank you for your consideration.

Sincerely,


John L. Friedman

Non-Self-Dual Nonlinear Gravitons

Philip B. Yasskin

James A. Isenberg*

Department of Mathematics
University of California
Berkeley, California 94720

*Chaim Weizmann fellow

Summary

Penrose has given a twistor description of all self-dual complex Riemannian spacetimes. We modify his construction to characterize all complex Riemannian spacetimes and all complex teleparallel spacetimes. This construction may be useful in finding non-self-dual solutions to the gravitational field equations (Einstein's or otherwise) without or with sources. It may also lead to a nonperturbative method of computing path integrals. Whereas Penrose shows that a self-dual spacetime may be specified by a deformation of projective twistor space (the set of α -planes in complex Minkowski space), we find that a Riemannian or teleparallel spacetime may be described by a deformation of projective ambitwistor space (the set of null geodesics in complex Minkowski space).

One of the most interesting recent developments in the theory of gravity has been Roger Penrose's translation of the problem of finding self-dual spacetimes from the language of differential geometry into that of algebraic geometry (i.e., "twistors").¹ This permits one to obtain a large class of spacetimes which satisfy Einstein's equations, without having to work directly with nonlinear partial differential equations. In this essay, we remove the self-duality restriction on Penrose's translation, and thereby pave the way to obtaining the general solution of Einstein's equations.

There *are* some virtues to the self-duality condition. A spacetime with metric g is called self-dual (or right-flat, or left-handed) if its curvature satisfies²

$$R^a{}_{bcd} = \frac{i}{2} \epsilon_{cdef} R^a{}_{b}{}^{ef} . \quad (1)$$

Taking the trace of this equation, we find that the Ricci tensor vanishes; hence the self-dual spacetimes automatically satisfy the vacuum Einstein equations. They are, therefore, potentially very useful as "gravitational instantons"³ about which to do perturbations in a Euclidean path integral formulation of quantum gravity. Further, as Penrose emphasizes in his paper "The Nonlinear Graviton",¹ these self-dual spacetimes are likely candidates for the one-particle states in some (as yet undiscovered) quantum theory of gravity.

There are, however, several reasons why the self-duality condition is overly restrictive and why one might want to generalize Penrose's construction to spacetimes which are non-self-dual (neither self-dual nor anti-self-dual). Firstly, it follows from equation (1) that any self-dual spacetime necessarily has complex curvature, and so it can

not represent a Lorentz signature metric on a real manifold. This precludes their use in classical spacetime physics. Secondly, since all self-dual spacetimes satisfy the vacuum Einstein equations, they cannot contain any sourcefields. This limits their use in quantum interaction physics. Thirdly, to build a valid quantum theory of gravity, one may find it necessary to modify the gravitational field equations (for example, by adding quadratic curvature terms to the Lagrangian, or by adding torsion to the connection). One would then want to construct solutions to these new equations which are not necessarily Ricci-flat, let alone self-dual. Finally we note that to perform a nonperturbative calculation of a path integral, it should be necessary to sum over all spacetime geometries, and not just those which satisfy the Einstein (or any other) field equations.

These physically restrictive aspects of the self-duality condition have led us to seek, and to find, a modification of the Penrose correspondence so that we have a twistor description of all complex Riemannian spacetimes.^{4,5} Indeed (for reasons which will become clear shortly), we obtain these through a twistor description of an even larger class of spacetimes--those which are "teleparallel".⁶ A teleparallel spacetime is a manifold M with a metric g and a connection Γ which is metric compatible ($\nabla g = 0$) and flat ($\hat{R}^a_{bcd} = 0$, where \hat{R}^a_{bcd} is the curvature of Γ). Thus, unless g is also flat, Γ must have torsion. The name teleparallel was chosen because the vanishing of \hat{R}^a_{bcd} says that (in a simply connected region) parallel transport according to Γ is path-independent so that there is a global notion of parallel.

We emphasize that every (parallelizable) complex Riemannian spacetime is contained in the class of teleparallel spacetimes. This is true in the sense that, given (M,g) , we may construct a corresponding teleparallel (M,g,Γ) by picking any framefield which is orthonormal according to g , and then defining Γ to be that connection relative to which the framefield is everywhere parallel. The Γ curvature then vanishes, while the metric curvature remains that of (M,g) . Thus, although our construction actually produces a twistor description of all teleparallel spacetimes, it includes a twistor description of all Riemannian spacetimes.

Before discussing the twistor objects which we have proven are in one-to-one correspondence with the teleparallel spacetimes, we recall how Penrose's nonlinear graviton correspondence works. For each complex Riemannian self-dual spacetime M which is a deformation⁷ of complex Minkowski space M , he shows, there corresponds a unique deformation PT of projective twistor space PT which preserves certain additional structures (to be described below). To understand PT and PT and to see how these relate to M and M (why, for example, a given M contains enough information to construct a unique PT , and vice-versa) it is useful to consider the "totally null planes" in an arbitrary complex spacetime. These are the 2-complex dimensional, totally geodesic surfaces on which every tangent vector is null. A straightforward calculation shows that every totally null plane is spanned by either a self-dual bivector or an anti-self-dual one. Those spanned by anti-self-dual bivectors are called α -planes while the self-dual ones are called β -planes.

In M , the set of α -planes is parametrized by PT (which mathematically is 3-dimensional complex projective space CP^3), while the set of β -planes is parametrized by dual projective twistor space PT^* (which is also CP^3 , but with the opposite complex structure).

In a curved spacetime M , however, there may or may not be any totally null planes. In fact, a given spacetime is self-dual iff (i) the α -planes exist, (ii) there are as many α -planes in M as there are in M (3-complex dimensions' worth), and (iii) the connection provides a path independent notion of whether any given pair of α -planes are parallel. Thus, for a given self-dual M , the space PT is defined to be a parameter space for the set of α -planes and may be shown to be a deformation of PT . This space PT carries two additional structures which are induced by M . First, there is an equivalence relation on PT which says that two points of PT are equivalent if the corresponding α -planes in M are parallel. This makes PT into a fibre bundle over the space D of α -plane orientations. Second, on each fibre of PT , there is a volume element μ .

The nonlinear graviton correspondence is one-to-one; thus there is a converse construction. That is, if one is given a space PT which is a deformation of PT , if PT is a fibre bundle over D , and if a volume element μ has been chosen on each fibre of PT , then as Penrose shows, one can reconstruct the spacetime M . The equivalence relation is used to construct the self-dual conformal geometry, while the volume element μ is used to fix the scale of the metric.

Since, as just noted, spacetimes which are non-self-dual do not contain a full complement of totally null planes, we need something else on which to base a twistorial description of such spacetimes. All

complex Riemannian spacetimes (with metric-compatible connection) *do* contain a 5-dimensional set of null geodesics; so we focus on them.

In complex Minkowski space, the set of null geodesics is parametrized by projective ambitwistor space PA , which is a five-complex dimensional hypersurface⁸ in $PT \times PT^*$. Just as with PT , there is an equivalence relation on PA which says that two points in PA are equivalent if the corresponding null lines in M are parallel. This equivalence relation defines a projection map $\pi: PA \rightarrow E$ so that

PA is a fibre-bundle over the space E of null directions in M .

In an arbitrary complex spacetime M with metric-compatible connection Γ , the set of null geodesics of Γ is parametrized by a manifold PA which we find is always a deformation⁷ of PA . If, like PA , the "deformed ambitwistor space" PA is a fibre-bundle over E , then in M there must be a path-independent notion of whether or not any given pair of null geodesics is parallel. A spacetime with such a property is teleparallel. Hence we conclude that there is a one-to-one correspondence between, on the one hand, deformations PA of PA which are bundles over E ; and on the other hand conformal equivalence classes⁹ of complex teleparallel spacetimes M .

To complete the twistor description of teleparallel spacetimes, we need a twistorial object which can distinguish among conformally equivalent spacetimes. But to define this object, we must first describe some additional structure in PA : Corresponding to each point p in M , there is a unique holomorphic section s_p of the bundle PA (i.e., $s_p: E \rightarrow PA$). The image $\text{Im}(s_p)$ of s_p in PA corresponds to the set of null geodesics in M which intersect at p . Since $\text{Im}(s_p)$ is a subset of PA , we may consider the restriction of the

tangent bundle $T(PA)$ to $\text{Im}(s_p)$. As a subset of this $T(PA)|_{\text{Im}(s_p)}$ we define $VT(PA)|_{\text{Im}(s_p)}$, the vertical tangent bundle at $\text{Im}(s_p)$, which consists of all $X \in T(PA)|_{\text{Im}(s_p)}$ such that $\pi_* X = 0$. It turns out that the extra structure we need to fix the conformal factor is a set of 1-forms $\{\sigma_p\}$, one for each section s_p , acting only on the vertical vectors:

$$\sigma_p: VT(PA)|_{\text{Im}(s_p)} \rightarrow \mathbb{C}.$$

Each 1-form σ_p determines the value of the conformal factor at the point $p \in M$, but not uniquely (i.e., more than one σ_p may determine the same conformal factor). Therefore, to get a unique twistorial object corresponding to the conformal factor, we demand that each σ_p must satisfy further conditions. Recall the definition of PA , which implies the existence of an imbedding $\acute{\iota}: PA \rightarrow PT \times PT^*$. It follows that the volume element μ on the fibres of PT which is preserved in the Penrose correspondence can be pulled back to PA , as can its counterpart $\tilde{\mu}$ on PT^* . We thus have a pair of 2-forms, $\nu := \acute{\iota}^* \mu$ and $\tilde{\nu} := \acute{\iota}^* \tilde{\mu}$ defined on PA . In deforming PA into $\mathcal{P}A$, both ν and $\tilde{\nu}$ are automatically preserved, up to scale. Thus we can define K to be the set of all one-forms λ on PA which satisfy¹⁰ $\lambda \wedge \nu = 0$ and $\lambda \wedge \tilde{\nu} = 0$. Among a set of σ_p 's which specify a particular conformal factor, there is a unique σ_p which lies in K . Thus we get the following result:

There is a one-to-one correspondence between

- (1) Teleparallel spacetimes M which are deformations of complex Minkowski space M

and

- (2) Deformations PA of projective ambitwistor space PA such that PA is a bundle over E with a 1-form

$$\sigma_p : VT(PA) \Big|_{\text{Im}(s_p)} \rightarrow \mathbb{C} \quad \text{specified for each}$$

cross-section $s_p : E \rightarrow PA$; demanding further that $\sigma_p \in K$.

Notice that this correspondence says nothing about whether the spacetimes M satisfy any field equations (apart from the teleparallel condition which, we have emphasized, in no way excludes any complex Riemannian spacetime). For some purposes, this is a virtue. Indeed, our twistorial characterization of non-self-dual spacetimes, with its separate encoding of the conformal geometry into PA and of the conformal factor into the choice of σ_p , seems especially well-suited for some formulations of a quantum field theory of gravity. Hawking, for example, has suggested that the quantum path integral over all spacetime geometries should be broken up into an integral over all conformal geometries and an integral over all conformal factors. This may be accomplished by first integrating over all deformations of PA and then integrating over all choices of σ_p .

Still, one would like to find a twistorial version of gravitational field equations. One way to seek this would be to translate well-known field equations (such as Einstein's) from spacetime language into that

of PA , σ_p , etc., via the correspondence. It is then quite possible that the tools of algebraic geometry would enable us to find many interesting solutions which have not been obtained by working directly with the differential equations on M .

An alternate approach, more in line with the spirit of the "twistor programme" of Penrose, would be to focus on the twistor side of the correspondence and seek (physically reasonable) field equations which appear most natural in terms of PA and σ_p . We note two such possibilities: The first is suggested by our experience with Yang-Mills theory.¹¹ There one finds that a general Yang-Mills connection corresponds to a fibre bundle over PA ; while the condition that the connection satisfy the Yang-Mills equations corresponds to the vanishing of the lowest order obstruction to extending that bundle from PA to a neighborhood of PA regarded as a hypersurface in $PT \times PT^*$. Similarly, the gravitational field equations may be the vanishing of the lowest order obstructions to extending the deformation of PA and the choice of σ_p from PA to a neighborhood of PA within $PT \times PT^*$. We do not yet know what differential equations in spacetime correspond to these conditions. They may be the field equations for Einstein's theory or some other metric theory, since any metric theory may be regarded as a teleparallel theory in which the teleparallel connection is ignored. Or they may be the field equations for some truly teleparallel theory.⁶

A second possible conjecture, involving only σ_p , may lead to a field equation for the conformal factor of M . One can demand that each 1-form σ_p be the restriction to $\text{Im}(s_p)$ of a globally-defined

1-form $\sigma: VT(PA) \rightarrow \mathbb{C}$. Again, we do not yet know the corresponding differential equation in spacetime, but we *do* know that this condition picks out at most one teleparallel spacetime from each conformal equivalence class (thus fixing the conformal factor). This is also a property of the equation $R = \text{const}$, where R is the scalar curvature of the metric g .

Whether or not these conjectures prove to be useful, we believe that it should not be too difficult to re-express any physically reasonable set of (metric or teleparallel) gravitational field equations in twistorial language using our correspondence. It may then be possible to use the tools of algebraic geometry to find spacetime solutions to these field equations.

Footnotes

1. The twistorial description of self-dual spacetimes is set up in Penrose's papers on the "Nonlinear Graviton" in Penrose, R., (1976), GRG 7, 171; and Penrose, R., (1976), GRG 7, 31. A large class of self-dual spacetimes is actually constructed using this correspondence in Ward, R., (1978), Proc Roy Soc. A 363, 289.
2. The totally antisymmetric tensor ϵ used in this essay is that for a Lorentz signature metric. For Euclidean signature, replace $i\epsilon$ by the Euclidean ϵ .
3. See, e.g., Gibbons, G. W., and Hawking, S. W., (1977), Phys Rev. D 15, 2752.
4. A "complex Riemannian spacetime" consists of a 4-complex dimensional manifold M and an analytic (not Hermitian) metric g . The metric-compatible and torsion-free connection $\{ \}$ is then uniquely determined in the usual form.
5. Our construction, like that of Penrose, only produces the geometry in a normal neighborhood.

6. Teleparallel geometries are also called fernparallel geometries or geometries with an absolute parallelism. Theories of gravity based on teleparallel geometries were studied originally by Einstein, Cartan, and Weitzenbach, and more recently by Møller and Hehl, Ne'eman, Nitsch, and von der Heyde. [See the article by Hehl, Nitsch, and von der Heyde in Held, A. (ed.), (1980), General Relativity and Gravitation, Plenum Press, N.Y., for a review which includes a fairly complete list of references.] In fact, the PPN expansion for the teleparallel theory proposed by Hehl et al. agrees with that for the standard Einstein theory up to fifth powers of the velocity; so that the theory agrees with all present solar system experiments.
7. Object A is a deformation of object B if there is a 1-parameter family of objects $C(t)$ such that $C(0) = B$ and $C(1) = A$.
8. PA is the quadric $Z^\alpha W_\alpha = 0$ in $PT \times PT^* = CP^3 \times CP^3$. The embedding is obtained by recognizing that every null line in M lies in a unique α -plane and in a unique β -plane and is precisely their intersection.
9. Two teleparallel geometries are conformally equivalent if their everywhere parallel orthonormal framefields are proportional. The square of the proportionality factor is the conformal factor relating the metrics.

10. Note that the undetermined scales of ν and $\tilde{\nu}$ are irrelevant in defining K .
11. See Isenberg, J., Green, P., and Yasskin, P., (1978), Phys Lett. 78B, 462; and Witten, E., (1978), Phys Lett. 77B, 394.