A NEW CLASS OF IDEAL CLOCKS

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SUMMARY

We have found a new class of ideal clocks within general relativity. They are self-gravitating systems such as rotating stars, rotating black holes and binary star systems. The gravitational redshift of the observed period of rotation of such clocks in a given, weak external gravitational field is shown to be the same as that of an ideal atomic clock. Because the clocks have structure and dynamics determined by gravitational interactions, the full non-linear machinery of general relativity must be used. This result is important for the binary pulsar PSR 1913+16, where the gravitational redshift of the pulsar's frequency caused by the companion's gravitational field is an observable effect.

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Ideal clocks play a fundamental role in special relativity and general relativity. Together with "rigid rods", they are the basic tools for "taking the measure" of spacetime. For example, an ensemble of identical ideal clocks, suitably synchronized, can be used to lay down a time coordinate in spacetime, while an individual ideal clock moving along some spacetime trajectory measures proper time along that trajectory. The hallmark of an ideal clock is that its rate as measured in a local, momentarily comoving, inertial frame must be independent of its motion through or its location in an external gravitational environment. Thus, for example, a pendulum clock on a rocket ship is not an ideal clock, since its rate depends directly on the acceleration of the ship relative to a local inertial frame. On the other hand, an atomic clock is an ideal clock, since the spacings of atomic or nuclear energy levels, whose values determine the frequency of an emitted electromagnetic quantum, are independent of all but the most extreme tidal gravitational fields, such as might occur near a spacetime singularity (see Misner, Thorne and Wheeler [MTW] 1973, §16.4 for discussion). In fact, any clock whose structure is determined strictly by the non-gravitational laws of physics is a candidate for an ideal clock, since, according to the Einstein equivalence principle (EEP), the non-gravitational laws of physics take their standard special relativistic forms in a local inertial frame; therefore the structure and the frequency of such a clock <u>will</u> be independent of the surrounding gravitational environment (see Will 1981 for detailed discussion of EEP).

An important observable consequence of this is the gravitational redshift: if two identical ideal clocks are placed at different locations in a static, weak gravitational field with potential $U(\underset{\sim}{x})$, then the frequency or wavelength shift Z, defined by $Z \equiv \Delta\nu/\nu \equiv (\nu_{rec} - \nu_{em})/\nu_{em}$, has the value

$$Z = \Delta U/c^2 = [U(\underset{\sim}{x}_{rec}) - U(\underset{\sim}{x}_{em})]/c^2 \quad ,$$

where the subscripts "em" and "rec" denote the values at the event of emission and reception of the signal that connects the two clocks. If for example the gravitational field is provided by a distant body of mass m, then $U = Gm/R + O(R^{-2})$, and if the receiving clock is at "infinity", then

$$Z = -Gm/Rc^2 \quad . \tag{1}$$

There are numerous different derivations of the gravitational redshift. The popular "inertial frame" derivation makes use of EEP and the static nature of the gravitational field to argue that if the frequency of a given type of atomic clock is the same when measured in a local, momentarily comoving, inertial frame, independent of the location or velocity of that frame, then the comparison of frequencies of two clocks at rest at different locations boils down to a comparison of the velocities of two local inertial frames, one at rest with respect to one clock at the moment of emission of its frequency-determining signal, the other at rest with respect to the other clock at the moment of reception of the signal. The frequency shift is then a consequence of the first-order Doppler shift between the frames (see for example §38.5 of MTW). In none of these derivations does the structure of the clock play any role whatsoever. This is because EEP makes a clean separation between gravitational and non-gravitational interactions, the former disappearing (modulo tidal effects) in local inertial frames.

There is another class of clocks — self-gravitating clocks — whose structure, by contrast, is strongly influenced by internal gravitational fields. Examples are rotating stars, rotating black holes, binary star systems and planetary orbits. Unlike atomic clocks, the periods or frequencies of these systems are not determined by fundamental constants; however, like atomic clocks, their frequencies can be stable to some desired degree of accuracy, thus once they have been calibrated appropriately, they qualify completely as clocks.

But are they ideal? More particularly do they experience the same gravitational redshift as atomic clocks? Unfortunately, EEP cannot guide us to an answer since it applies only to non-gravitational systems. The structure of a gravitational clock is determined by general relativity, and so the full, non-linear machinery of that theory must be used to answer such a question. Although one might expect the outcome to be the standard redshift as expressed in Eq. (1), the result is not guaranteed. In fact, since the clock itself has self-gravitational interactions, it produces its own gravitational redshift, for example, of the frequency of an atomic clock located at its center. The internal gravitational effects that produce this "central" redshift could, via some non-linear interactions with the field of the external body, modify the final redshift of the gravitational clock, so that instead of Eq. (1), the shift might take the form

$$Z \simeq -\alpha(Z_c)Gm/Rc^2 \quad , \qquad (2)$$

where α is some function of the central, atomic-clock redshift Z_c of the system. The more relativistic the system, the larger the central redshift; for a system of mass M and characteristic size d, Z_c is given roughly by

$$
\begin{aligned}
Z_c &\sim -GM/dc^2 \\
&\sim 10^{-8}(M/M_\odot)(1 \text{ a.u.}/d) \quad , \quad \text{[close binary system]} \\
&\sim 10^{-1}(M/M_\odot)(10 \text{ km}/d) \quad , \quad \text{[neutron star]} \\
&\sim \text{undefined} \quad . \quad \text{[black hole]}
\end{aligned}
\qquad (3)
$$

Thus the nature of the gravitational clock could in principle affect the gravitational redshift of its frequency.

Actually, this is more than just a matter of principle, it could have important observable consequences, namely in the binary pulsar PSR 1913+16. The pulsar's period is affected by the gravitational redshift produced by its companion and by the second-order Doppler shift produced by its orbital motion, both effects being variable because of the highly eccentric orbit. By measuring the variations in the pulsar period (more specifically in the pulse arrival times), observers have made the first accurate determination of the mass of a radio pulsar. The result is $1.42 \pm 0.06\ M_\odot$ (Taylor and Weisberg 1982) a value remarkably close to the Chandrasekhar mass limit for a degenerate neutron core. A twenty per cent deviation from unity of the parameter $\alpha(Z_c)$ in Eq. (2) would produce a ten per cent change in the pulsar's mass value.

In fact, such a modification is not necessary, for we have found that the redshift of a gravitational clock is independent of its structure, and is given by

$$Z = -Gm/Rc^2 \quad , \tag{4}$$

for an observer at infinity. Therefore, general relativistic clocks are ideal.

As we have remarked before, a simple, universal derivation of this fact, of the kind available for atomic clocks in EEP, is not possible here. Instead we have considered three specific examples and used the full machinery of Einstein's equations together with appropriate approximations (details will be published elsewhere).

(i) Rotating Relativistic Star: This is the closest idealization of the binary pulsar. We considered such a star in the gravitational field of a distant body of much smaller mass, modelled for our purposes as a non-rotating spherical shell. We then used a simple matching procedure together with a change of variables to show that, as a function of the distance of the external body, the uniform angular velocity of the star as observed at infinity is given by

$$\Omega \simeq \Omega_0 (1 - Gm/Rc^2) \quad , \tag{5}$$

where Ω_0 is a function of the angular momentum, total baryon number and other intrinsic parameters of the rotating star, and is independent of R. Then

$$Z_{star} \equiv (\Omega - \Omega_0)/\Omega_0 = -Gm/Rc^2 \quad . \tag{6}$$

This result is independent of the compactness of the star and of its angular velocity.

(ii) Slowly Rotating Black Hole: We considered a slowly rotating black hole perturbed by a stationary axisymmetric ring of matter. By extending previous results of Will (1974,1975), we showed that the observed angular velocity of the horizon ω_H (angular velocity of the generators of the horizon as seen from infinity) is given by

$$\omega_H = \frac{1}{4}\left(c^4/G^2\right) J_H \, M_{ir}^{-1}\left(1 - Gm/Rc^2\right) \quad , \tag{7}$$

where J_H and M_{ir} are the constant angular momentum and irreducible mass of the black hole. Then

$$Z_H = -Gm/Rc^2 \quad . \tag{8}$$

This example represents the extreme in compactness, yet yields the same redshift.

(iii) Binary Systems: Working in the post-Newtonian limit of general relativity, we considered a binary system in the gravitational field of a distant third body of much smaller mass. Initially, the binary system was in a circular orbit, while the relative orbit of the binary system and the third body was elliptical (in the sense of osculating orbit elements). We then calculated post-Newtonian corrections to the observable "mean longitude" of the binary system (Brouwer and Clemence 1961). The rate of change of this quantity defines the orbital period as measured at infinity: the result, modulo constants is

$$P = P_o\left(1 + Gm/Rc^2\right) \quad , \tag{9}$$

where m is the mass of the third body. The resulting redshift is thus

$$Z_{binary} = \left(P^{-1} - P_o^{-1}\right)/P_o^{-1} = -Gm/Rc^2 \ . \tag{10}$$

Thus we have found a new class of ideal clocks—self-gravitating general relativistic clocks. Although these clocks are unlikely to see the same everyday use as do atomic clocks, their importance as probes of relativistic gravity in systems such as the binary pulsar is assured. Furthermore, their existence is another of several manifestations of the elegance and simplicity of general relativity.

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