# SCALAR GRAVITATION


**Summary**

Nordström's scalar theory of gravitation is discussed from a modern point of view, and compared with some aspects of general relativity. We discuss the equations of motion for test masses, the field equations in the presence of matter, the extent to which the principle of equivalence, Mach's principle, and the expansion of the universe are contained in this model. The theory implies what amounts to a Riemannian metric. The question is considered whether this model leaves any room for antigravity.



Marcel Wellner
Indiana University
Bloomington, Ind.
+
Guido Sandri
A.R.A.P.
Princeton, N.J.

# Scalar Gravitation

Marcel Wellner, Indiana University, Bloomington, Indiana

and

Guido Sandri, A.R.A.P., Princeton, New Jersey

## 1. Introduction

This article will be devoted to a scalar model of gravitation. The theory to be discussed is due to Nordström[1] and dates back to 1913. It will not be presented as a superior alternative to general relativity. There are, however, many good reasons, other than historical,[2] for exhuming it. It is definitely the simplest model which may be said to adapt Newton's law of gravitation to the requirements of special relativity. As a consequence it provides, from the conceptual and pedagogical points of view, a natural stepping stone toward the study of general relativity itself.

This view is reinforced by some remarkable features of this model, some of which, to the best of our knowledge, are brought up here for the first time.

a) If, following its prescriptions, one introduces a Newtonian gravitational field in Minkowski's flat 4-space, one finds that measuring instruments (i.e. clocks and rods) do not obey the Euclidean rules: Riemannian space, which is pushed out the door, comes back through the window.

b) The principle of equivalence holds in its full meaning, i.e. not only in the sense of Galileo, but also in the sense of Pound and Rebka.[3]

c) It embodies Mach's principle to a remarkable degree: the ratio of gravitational to inertial mass depends on distant surrounding masses.

d) It "predicts" Hubble's red shift.

With the advent of general relativity, Nordström's theory was soon abandoned, even by its originator, for reasons which were largely aesthetic, or, as put more accurately by Einstein,[4] because the invariance group of general relativity is so much larger. Further successes of general relativity contributed, sometimes for the wrong reasons, to push Nordström's theory into the shadow. For example, the latter theory implies a perihelion procession which is only one sixth as large as in the former and retrograde at that. Also, it implies no net gravitational deflection of light. Observational evidence regarding those two small effects is well known to favor general relativity. Owing to the latter's prestige, however, there has been a somewhat unhealthy lack of searching for alternative causes. The evidence for another radical theory, quantum mechanics, had to be considerably stronger, to say the least, before competing theories were definitively laid to rest. As far as the gravitational red shift is concerned, it is now widely admitted to provide no crucial test for general relativity. This is explicitly illustrated by feature b) above. That feature, as well as c) and d), appears to have been unrecognized so far, another cause for neglect.

An often-heard objection against a scalar gravitational field is the fact that it can only be coupled to the trace $T_\mu{}^\mu$ of the stress-energy tensor, whereas observation unambiguously points to $T^{00}$ . This objection was shown by von Laue[5] to be fallacious, as Einstein[4] himself recognized. A final argument against scalar theorie must be mentioned: in such theories, if electromagnetic radiation is enclosed in a box, its coupling with the gravitational field occurs at the walls only. Hence if a reflecting tube with square cross-section contains two weightless reflecting pistons which enclose a cubic cavity with radiation between them, and if those pistons are rigidly connected to each other but can slide in the tube (see Fig. 1), then the weight of the enclosed radiation, as measured by the work done in sliding the pistons, will only be a third of what it would be in a rigid cube of the same size.

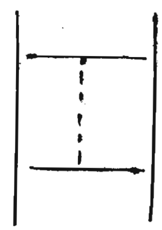

Fig. 1

This seems absurd. This argument, adduced by Einstein,[4] cannot be considered convincing until supported by a careful analysis of edge effects at the points of contact between wall and moving piston. This must be done owing to the extremely singular nature of the gravitational coupling in this case.

## 2. Equations of Motion for a Test Mass

Newton's second law

$$\ddot{x}^k = \partial^k \varphi \qquad (k = 1,2,3), \tag{1}$$

where $\varphi$ is the gravitational potential, is the nonrelativistic limit of

$$\ddot{x}^\mu = \partial^\mu \varphi - \dot{x}^\mu \dot{x}^\nu \partial_\nu \varphi$$

$$(\mu, \nu = 0,1,2,3;\ c=1). \tag{2}$$

This equation is Lorentz covariant if $\varphi$ is a scalar and if the dot denotes differentiation with respect to the proper time $d\tau = (dx^\mu dx_\mu)^{1/2}$.

In eq. (2) the additional term, which becomes negligible at low speeds, is required in order that the motion be not overdetermined by the fact that there are four instead of three equations. Contraction of both sides over yields an identity. Setting

= , (3)

we obtain what we shall call the standard form

. (4)

For fields encountered in practice (e.g. at the surface of astronomical bodies),

. (5)

It should be noted that (4) may be obtained from the variational principle

=0. (6)

It would, however, be premature to talk about a Riemannian metric before investigating the mode of production of by matter. Throughout the discussion we shall, for conceptual clarity, use the formal — one might even say Platonic — flat-space coordinates , in terms of which the theories are postulated to be Lorentz-invariant. Actual instruments may or may not indicate the , as we shall see below.

## 3. Field Equations in the Presence of Matter

We must generalize Poisson's equation

, (7)

where G is the gravitational constant in rationalized units, and where is a static mass distribution. If is time-dependent, the left side becomes , or, for low fields, (see (5)). In order to make the right side a scalar, we invoke von Lane's theorem according to which, for a closed system whose center of mass is at rest, we have

$$= \quad , \qquad (8)$$

provided the right side is interpreted as a time-average. There is now no objection against writing

$$= \qquad (9)$$

in the limit of low fields. The choice of the exact equation is governed by the conservation of energy and momentum. Consider for simplicity all matter to be "dust", i.e. a massive fluid having no internal stresses other than gravitational. Let be the proper mass density of dust (i.e., measured when travelling along with the fluid). Consider the one-parameter family of field equations

$$= \quad , \qquad (10)$$

being an arbitrary number. If the boundary condition

$$(11)$$

at large distances is observed, all these equations are Newtonian in the appropriate limit. If one uses (4), (10), and the "conservation of dust"

$$= 0, \qquad (12)$$

then it is readily verified that

$$(13)$$

satisfies

. (14)

There remains the problem of fixing .

## 4. The Principle of Equivalence

Nordstrom showed that is the correct choice if the Galilean principle of equivalence (i.e., the equality of gravitational and inertial masses) is to be satisfield not only for test masses but also for macroscopic bodies. Given a quasi-static spherical distribution of finite extent, one can determine its gravitational mass by solving (10) for at large distances, thus obtaining

(15)

for some determined by . Similarly, its inertial mass may be determined by

, (16)

being given by (13). It then turns out that = requires = 0.

Here we wish to present totally different ( and perhaps more instructive) considerations leading to the same result. First we show that clocks and rods are affected by the presence of . To construct (ideally) a standard of length it is enough to choose a reference mass , and to note that the theory then contains a characteristic length = . (The scale of is provisionally fixed by (11).) A characteristic time is provided by . Next we shall assume that we artificially modify the scale of in a certain region, for example by enclosing it in a hollow massive shell, thereby changing the effective boundary condition. In that region let the consequent change be

( const.)

Then the new physical laws may be obtained from the old ones by the transformation

.

This is nothing but the gauge group under which the theory is invariant. This shows in particular that the characteristic time intervals T of a clock will be affected by a change such that

= . (17)

(A very careful argument is needed in order not to get the wrong sign here!) Standards of length will be affected in the same way, so that the speed of light is unchanged: the deformations are conformal in space-time. It is well known that equivalence requires

. (18)

a behavior which has been strikingly confirmed by using the Mossbauer effect. Hence one obtains .

Even the apparent local bending of a light ray by gravitation may be obtained in this way. Using our Platonic coordinates, we obtain for a "rigid" rectangular frame in a vertical downward field the conformal deformation illustrated in Fig. 2. In these same

Fig. 2

coordinates, light is not affected by gravitation, since = for radiation in empty space. (Alternatively, it may be verified that the trajectory of a particle approaches a straight line as its speed approaches that of light.) Hence the path of a light ray (dotted line in Fig. 2 b ) will seem to be curved downwards. For the same reasons light rays, in this model, suffer no <u>net</u> deflection under the gravitational attraction of the sun.

## 5. <u>The "Expansion" of the Universe</u>.

Hubble's red shift is easily accounted for if one assumes a uniform static non-zero density of matter. In a frame of reference which is stationary with respect to that matter, we have from (10)

= , (19)

under the assumption that the universe is uniform in space:

. (20)

With a suitable origin for one has

const. (21)

Suppose the present date is negative. Then is decreasing. Emission of the observed light occurred at lower and hence the light is seen as if under a gravitational red shift, which should be proportional to the distance if the latter is not too large. The situation is illustrated in Fig. 3, in which is the present time. The slope of the curve at is a measure of Hubble's constant.

Fig. 3

## 6. Mach's Principle

The ratio of inertial to gravitational mass is affected by , since this is the coefficient of in (4). The preceding section shows that, if the mean density of matter is non-zero, the boundary condition $\phi \to 1$ is a property of the present time only and hence that the ratio of inertial to gravitational mass changes with the age of the universe. The special value = 1 is of course due to a short-sighted choice of units. More generally, if the effective boundary condition is modified, say by enclosure in a massive shell, the above ratio will be affected through the inertial term in (4).

## 7. Antigravity

For a single point-source, the solution (15) is exact. Therefore = 0 at a critical radius (corresponding to the Schwartzschild radius of general relativity)

,

inside which . Again in view of eq. (4) this means a reversed inertia, and hence an acceleration away from the central mass. A similar situation becomes universal before $t_1$ and after $t_2$ in Fig. 3. However, these situations must be considered unphysical, as they cannot be reached in a physically continuous manner from the more usual situations; the metric must become singular at the transition points. This, in fact, is precisely where the Cartesian *Ansatz* fails us, and where curvilinear coordinates would become necessary.

**Acknowledgments**

Many of the present considerations arose when the authors were at the Institute for Advanced Study, Princeton, New Jersey. They are grateful to Professor J.R. Oppenheimer for his hospitality. They also wish to acknowledge some stimulating discussions with Dr. Hans Buchdahl and Professor R.H. Dicke. One of us (M.W.) is grateful to the National Science Foundation for a postdoctoral fellowship.

**Footnotes**


1) G. Nordström, Ann. d. Phys. **42**, 533 (1913).

2) For historical considerations, see Sir E. Whittaker, A History of the Theories of Aether and Electricity, Vol. II, especially p. 153.

3) R. V. Pound and G. A. Rebka, Phys. Rev. Letters **4**, 337 (1960).

4) A. Einstein and M. Grossmann, Z.f. M. u. P. **62**, 225 (1913), especially § 7.

5) M. v. Laue, Jahrb. d. Rad. u. El., **14**, 263 (1917).