Gravitational Stability of Large Masses

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Summary

In the gravitational equilibrium of white dwarf stars, instability sets in at 1.44 x sun's mass for reasons based on special relativity. The radius of stars of nearly the limiting mass is determined by two predictions of general relativity that have never been subject to experimental test - gravitation of stress up to first order in the gravitational constant G, and curvature of space up to second order in G. It is suggested that the correlated distribution of masses and radii of stars of this kind will provide a new experimental test of general relativity.

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The theory of gravitation currently accepted, - the general theory of relativity - has confronted experiment in a rather small number and variety of situations. That is why we call attention here to a different kind of situation - the instability of very dense and massive stars. Others have discussed the motion of slight masses on the surface, or far outside, of massive bodies. Here we shall be concerned with the internal equilibrium of the massive bodies. Past discussions have been limited to the effects of weak gravitational fields, where the characteristic non-linearities of Einstein's theory have been relieved by expansion of all field quantities up to first order in the gravitational constant, G. In contrast, one prediction we shall draw from general relativity, namely, a minimum radius for a stable white dwarf star, relies on effects of second order in G.

All tests, including the indirect one proposed here, suffer from the limitation of experimental material to weak gravitational fields. The potentials of Newton or Einstein, for a static field, unlike electromagnetic potentials, have a quantitative meaning (under certain sensible limitations of coordinate system), and are large or small in co parison to an absolute standard, the square of the limiting speed $c = 3 \times 10^{10}$ cm/sec. A mass M of radius R has a strong field if $GM/Rc^2 \approx 1$. No known mass is dense or large enough. Since general relativity itself is the basis for the argument we shall give against the stability of such mass concentrations; the unavailability of strong static fields emerges here as a limitation in principle.

A static mass tends to be spherically symmetric, and needs two potentials, U_o and U_r, to describe it in general relativity, one controlling the red-shift of a resting clock and the other the contraction of a resting, radial ruler. These effects are symbolized by a line element

$$ds^2 = (1 + 2U_o)(cdt)^2 - (1 + 2U_r)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2. \quad (1)$$

The potentials are measured in absolute units c^2, and outside the mass reduce to the Newtonian value U_n

$$\text{(outside)} \quad U_o = U_r = U_n = -GM/rc^2. \quad (2)$$

The correctness of (2) has been tested by red-shift experiments where the effect is based entirely upon U_o, by the perihelion advance of close planets where two-thirds of the effect comes from U_o and one-third from U_r, and by the deflexion of light where U_o and U_r contribute equally. The general theory of relativity seems to be well confirmed by these experiments, but only to first order in the gravitational constant G.

Within a mass, the distribution of radial stress p_r, and of energy density ϵ, including the rest-energy of all masses, determine the potential U_r from

$$U_r = -(4\pi G/c^4)\left(\int_o^r r^2 \epsilon dr\right)/r, \tag{3}$$

essentially the virial of a Newtonian potential generated by the energy density. U_o is given by a more complicated expression which reduces to the Newtonian potential generated both by energy density and stress, when expanded up to first order in G:

$$\Delta U_o \approx (4\pi G/c^4)[\epsilon + (1/r^2)(d/dr)(r^3 p_r)]. \tag{4}$$

That all energy gravitates is exemplified in (3), and is known from experiments proving the equality of inertial and gravitational mass, since kinetic and interaction energy of nucleons within nuclei affect considerably the total inertial of matter. The equation U_o has been cast into the form (4) to show a less obvious consequence of the theory - gravitation of stress.

The remaining field equation of Einstein gives precise expression to this idea:

$$\frac{dp_r}{dr} + \frac{2}{r}(p_r - p_t) + \left(\frac{4\pi G}{c^4}\right)\frac{[\epsilon + p_r][\int_o^r r^2 \epsilon dr + r^3 p_r]}{r^2 (1 + 2U_r)} = 0 \tag{5}$$

Setting aside for the moment a possible anisotropy of stress expressed by $p_r - p_t$, (the difference between radial and tangential stresses), and neglecting effects of higher order in G than the first, (introduced by U_r in the denominator) we see in (5) how the pressure gradient (dp_r/dr) is balanced by the gravitational force from all matter closer to the center, $-(4\pi G/c^2)(1/r^2)\int_o^r r^2 \epsilon dr$ acting upon the density of matter, (ϵ/c^2), at that point - except that both force and density are increased by an amount proportional to p_r. This gravitational effect of stress together with the second-order effects from U_r in the denominator are the features of Einstein's theory leading directly to the existence of a minimum radius for white dwarf stars.

For massive bodies we turn to stars; but generally, they are not dense enough for our purpose. While a star burns, energy pouring out of its interior and meeting the opacity of matter as an effective resistance to its flow, inflates the star to relatively great size. Only when the most combustible nuclear fuels, hydrogen and helium, have been consumed, can the star cool and condense to the white dwarf state, which interests us here, as a gravitational laboratory.

The white dwarf, - a mass comparable to M_o, the mass of the sun, of a size comparable to that of the earth, - is a fascinating object from many other points of view. Here we can cite only two of its properties, discovered in theory by Chandrasekhar[(1)] and Landau[(2)] and experimentally verified in broad outline: The greater the mass, the smaller the radius; and, as the mass approaches a limiting value $\approx 1.44\ M_o$ the radius shrinks to zero, all greater masses being unstable.

At densities of tons or hundreds of tons to the cubic inch, all atoms form a single giant atom with some 10^{57} distributed nuclei, and with their electrons occupying joint energy levels in accordance with the exclusion principle. The nuclei gravitate towards the center pulling electrons after them by the strong electric force between their charges. But the electrons resist this confinement. According to quantum principles, the zero-point momentum of a particle varies inversley as the linear dimensions of its prison. The pressure it can supply, proportional to the mean product of speed and momentum, must therefore increase with concentration, and hence can balance gravitation at a point determined dimensionally as $R \propto M^{-1/3}$.

The second fact is a consequence of special relativity. As the mass is increased, the central density rises, and the electron's kinetic energy becomes relativistic. But as its speed approaches and sticks at c, its contribution to the pressure increases less rapidly with concentration, and eventually fails to supply an effective counterbalance to gravitation. Hence the existence of a limiting mass. Systems larger than this must either find some way of shedding their excess mass before they cool, or they must collapse suddenly. Unless relativistic collapse is associated with supernovae, it has never been observed.

Stable white dwarfs just smaller than the critical mass can be very small in size and can generate considerable gravitational potential.

Though the fields are still weak, the effects of gravitation of energy and stress and of non-linearity, (curvature of space), unite in enhancing the attractive forces, and therefore precipitate the instability described above at a slightly smaller mass. Most important, stars collapse from a finite value of the radius, which represents the minimum radius for a stable star of this type. The discovery of the minimum radius effect was made by Oppenheimer and Volkoff[3], working with the somewhat simpler system of an assembly of neutrons. They found a limiting mass of .78 M_o and a minimum radius of only 10 km. Kaplan[4] applied their theory to white dwarfs and deduced a limiting mass of 1.4 M_o and a minimum radius of 1100 km; but unfortunately his work was based on an incorrect theory of pressure and density under relativistic conditions.

In order to deal with systems of particles interacting by way of electric and gravitational forces under such conditions, the authors[5] developed a general relativistic form of statistical mechanics.

The electrostatic field plays an important part in this theory, entering by way of its potential V, field strength E and charge density ρ according to

$$E = -\left(\frac{1 + 2U_r}{1 + 2U_o}\right)^{1/2}\left(\frac{dV}{dr}\right), \qquad \frac{d}{dr}(r^2E) = -4\pi r^2\rho(1 + 2U_r)^{1/2} \qquad (6)$$

and contributing $(+E^2/8\pi)$ to ϵ and p_t, and $(-E^2/8\pi)$ to p_r. Particles of mass m and charge e contribute to ϵ and, (isotropically), to p_r and p_t in proportion to their distribution functions N(W'). The form of the latter as a function of energy W' is governed by ordinary statistical considerations, reducing to Maxwell or Fermi distributions in thermal equilibrium. The weights with which they are averaged are governed by special relativity; and only the relation of W' to the parameter of energy distribution W shows the characteristic effects of gravitation and general relativity:

$$\epsilon = \int_1^\infty W^2(W^2 - 1)^{1/2}\sum mc^2N(W')dW$$

$$p_r = p_t = \int_1^\infty (1/3)(W^2 - 1)^{3/2}\sum mc^2N(W')dW$$

$$\rho = \int_1^\infty W(W^2 - 1)^{1/2}\sum e\,N(W')dW \qquad (7)$$

$$W' = eV + mc^2W(1 + 2U_o)^{1/2}$$

With the aid of (7) there is now sufficient information to solve any problem of this kind. We will not dwell on the fairly intricate computations. It must suffice here to say, that in the limit of absolute zero temperature the relation between ε and p is exactly that of special relativity[1], as was assumed on fairly obvious grounds in the simple neutron case[3]. But the work of Kaplan is clearly contradicted. Though the correct computation gives nearly the same limiting mass, 1.37 $M_{\odot}$ the minimum radius of white dwarf is 2080 km, a figure more likely to impress astronomers.

Should we expect therefore that white dwarfs will never be found with smaller radii than our figure, thus leaving an empty quarter in the Hertzsprung-Russell diagram? Wheeler[6] has shown that inverse beta decay can intervene to aggravate gravitatational instability. As the system contracts, the energy of an electron at the center can be great enough to combine with a nuclear proton, converting it into a neutron and a low energy neutrino. The pressure will be relieved, at first, by replacing an energetic electron with a relatively less energetic, or bound, neutron. The system will therefore contract, providing more electrons at high energy to activate further beta decays. The star must evolve to a neutron assembly, there to reach equilibrium, or to undergo a second gravitational collapse from a limiting radius of about 10 km. Since beta processes are notoriously slow - especially inverse beta decay close to its threshold, we can expect stars to linger in these unstable forms for times not negligible on the astrophysical time scale, and hence to populate the "empty quarter" of the H-R diagram. On the other hand, the same reason gives general relativity more opportunity to influence the distribution of radii for stars close to the limiting mass.

In order to judge which features of the final distribution of masses and radii in the "empty quarter" are caused by general relativity, and which could be explained on the basis of inverse beta instability alone, it will be necessary to estimate the rates of evolution of dense stars under these contrasting conditions. The calculations in progress to-day, are still incomplete; but we expect, with confidence, that the gravitation of stress and the second order curvature will eventually confront experiment in this domain.

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