

Quantum Theory and Gravity

(Abstract)

Ryoyu Utiyama
Osaka University
Osaka, Japan

A new kind of approach to the unification of quantum theory with general relativity is proposed. For this purpose a local Lorentz-frame (or Vierbein system) together with a local Hilbert-space is attached to each world point in order to establish a Lorentz-covariant quantum theory at the neighbourhood of each world point. The connection between state vectors at different world points is derived from the connection of the corresponding Vierbeins at these points. This connection enables one to derive covariant equations in the macroscopic world from q-number equations in the microscopic world.

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Ryoyu Utiyama
Department of Physics
Osaka University
Osaka, Japan

As Wigner⁽¹⁾ pointed out, there seems to be no essential conflict between the quantum mechanical concepts and those inherent in the special relativity. Accordingly if a physical system is embedded in a gravitational field, one has to fix, first of all, in the space-time a special local frame of reference which locally eliminates the gravity at least in the vicinity of its origin. After removal of gravity one can set up a Hilbert-space following the conventional prescription. It should be noted that the relativistic quantum theory thus obtained allows us to have a definite interpretation only inside a small region of the space-time around the origin of the above mentioned frame of reference (let us call this a local Lorentz-frame or $L. L. F.$). Accordingly, if one wants to describe a quantum system which is located in the outside of the above mentioned region, one has to employ another $L. L. F.$ and another Hilbert-space associated with this new $L. L. F.$.

In order to make our argument more concrete let us introduce any system of coordinates ξ^μ ($\mu = 0, 1, 2, 3$) in the space-time. This system of coordinates is supposed to be a

macroscopic one and consequently it is assumed that one can fix this system without suffering from the quantum limitations of measurements. Let it be called the world coordinate-system.

Next let us define a neighbourhood $\omega(P)$ of a world-point P as follows; $\omega(P)$ is a small region of space-time surrounding the point P , in which one is able to eliminate the effect of gravity by choosing a skilful $L.L.F.$. Now suppose that the whole space-time is divided into a great number of small neighbourhoods. To each neighbourhood, say $\omega(\xi)$ at ξ , an $L.L.F.$ $L(\xi)$ is attached respectively in order to remove the gravity from $\omega(\xi)$. Finally let a Hilbert-space $\mathcal{H}(\xi)$ be defined at each $\omega(\xi)$ in terms of the quantities defined in reference to $L(\xi)$. Here it must be noted that if $L(\xi)$ is subjected to a local Lorentz-transformation, the basic vectors of $\mathcal{H}(\xi)$ should be also transformed by a corresponding unitary operator.

According to the line of thought so far stated, a state vector Ψ belonging to $\mathcal{H}(\xi)$ depends upon the coordinate ξ in addition to the other quantum variables. In the conventional theory, because of the flatness of the space-time, state vectors at different points and representing a same physical state are put equal to each other, namely, paralalled to each other. Thus in such a case Ψ is essentially independent of ξ and we need not distinguish Hilbert-space_s at different points from others.

On the contrary, if the world is curved, the Vierbein or L. L. F. at $\xi + d\xi$ is not same as that at ξ . The Vier bein $h_a^\mu(\xi)$ (Latin indices represent components with respect to L. L. F. and Greek ones the components of world tensors, both run from 0 to 3) at ξ is related with $h_a^\mu(\xi + d\xi)$ by

$$h_a^\mu(\xi + d\xi) = \left\{ h_a^\mu(\xi) - d\xi^\nu \Gamma_{\rho\nu}^\mu(\xi) \cdot h_a^\rho(\xi) \right\} \\ = d\xi^\nu A_{\nu, ab}(\xi) \cdot h^{b\mu}(\xi) \quad (1)^{(2)}$$

This relation shows that $h(\xi + d\xi)$ differs from the vector

$$h_{||}(\xi + d\xi) \text{ produced by a parallel transport of } h(\xi)$$

along some curve and that a further infinitesimal Lorentz-
transformation of $h(\xi)$ is necessary for getting the correct

$h(\xi + d\xi)$. If $A(\xi)$ vanishes around the point ξ , $h(\xi + d\xi)$ is identical with $h_{||}(\xi + d\xi)$. Consequently a state vector $\Psi_\alpha(\xi + d\xi)$ representing a state α can be chosen to be identical or parallel with a state vector $\Psi_\alpha(\xi)$ representing the same state. However, if $A(\xi)$ does not vanish, $\Psi_\alpha(\xi + d\xi)$ differs from $\Psi_\alpha(\xi)$ by a Lorentz-transformation which corresponds to the right-hand side of (1).

Therefore we have a relation

$$\Psi_\alpha(\xi + d\xi) - \Psi_\alpha(\xi) = \frac{i}{2} A_{\mu, ab}(\xi) \cdot d\xi^\mu \cdot M^{ab} \Psi_\alpha(\xi), \quad (2)$$

which shows a Lorentz-transformation of Ψ_α with an infinitesimal parameter $A_{\mu, ab}(\xi) \cdot d\xi^\mu$. Here $M^{ab} = -M^{ba}$ is a total

angular momentum operator defined inside the neighbourhood $\omega(\xi)$. M at $\omega(P)$ can be regarded as commutable with M at $\omega(Q)$ when Q is distant from P . It may be reasonable to assume further that M defined at two neighbouring points are almost identical with each other because of the continuous distribution of physical systems.

The equation (2) is in general not integrable. Accordingly in order to attach a Hilbert-space to each $L.L.F.$ or each neighbourhood, let us consider a bundle of curves emanating from some fixed point O and covering the whole world. If the $A_{\mu,ab}(\xi)$ is known everywhere, we can fix a state vector at any point ξ by integrating (2) along a particular curve passing through the point ξ with the given initial condition $\Psi_a(0)$.

Now our view-point is that, in each neighbourhood the conventional Lorentz-covariant quantum theory is valid and the effect of gravity is unnecessary for the description of the physical phenomena in one microscopic system. On the other hand when some data at some neighbourhood are to be compared with other data at another distant point, one has to take into account the effect of gravity. This effect is correctly involved in the classical equations of general relativity. Therefore one has to show how the classical general covariant equations in the macroscopic world can be derived from the quantum equations in the microscopic world.

Consider two $L.L.F.S$, $h_a^\mu(\xi)$, and $h_a^\mu(\xi+d\xi)$ at

two adjacent points $P(\xi)$ and $Q(\xi+d\xi)$, respectively. The local coordinate of the point Q viewed from the L.L.F. at P is given by

$$\chi_\alpha^k = h_k^\mu(\xi) \cdot d\xi^\mu.$$

Let us consider a vector operator $V^k(x)$ which is defined with respect to the L.L.F. at P . Since the point Q can be regarded as a point inside $\omega(P)$ and furthermore in $\omega(P)$ the space is free from gravity, a vector $V_{//}^k(Q)$, which is a parallel transport of $V^k(P)$, is given by

$$V_{//}^k(Q) = V^k(P). \quad (3)$$

The classical counterpart of V^k at the point P is, according to Ehrenfest,

$$V^\mu(\xi) = h_k^\mu(\xi) \left(\Psi_\alpha(\xi) \cdot V^k(x_P=0) \Psi_\alpha(\xi) \right).$$

In the same way, we have at Q

$$V_{//}^\mu(\xi+d\xi) = h_k^\mu(\xi+d\xi) \cdot \left(\Psi_\alpha(\xi+d\xi) \cdot V_{//}^k(x_Q) \Psi_\alpha(\xi+d\xi) \right) \quad (4)$$

Inserting (2) and (3) into (4) and neglecting terms of higher orders in $d\xi$, we have

$$V_{//}^\mu(\xi+d\xi) = V^\mu(\xi) + d\xi^\nu \left\{ \frac{\partial h_k^\mu}{\partial \xi^\nu} h_k^\rho \cdot V^\rho(\xi) - \frac{i}{2} h_k^\mu \cdot A_{\nu,ab} \left(\Psi_\alpha [M^{ab}, V^k(x=0)] \Psi_\alpha(\xi) \right) \right\}.$$

By definition we know

$$[M^{ab}, M^{rs}] = i \{ \eta^{ar} M^{bs} + \eta^{bs} M^{ar} - \eta^{as} M^{br} - \eta^{br} M^{as} \} \quad (5)$$

and

$$[M^{ab}, V^{k(0)}] = i \{ \eta^{ak} V^{b(0)} - \eta^{bk} V^{a(0)} \}, \quad (6)$$

where η is the Minkowskian metric. Accordingly $V_{//}^{\mu}(\bar{z} + d\bar{z})$ leads to

$$V_{//}^{\mu}(\bar{z} + d\bar{z}) = V^{\mu}(\bar{z}) + d\bar{z}^{\nu} h^{\rho}_{\nu}(\bar{z}) \left\{ \frac{\partial h^{\mu}_{\rho}}{\partial \bar{z}^{\nu}} + A_{\nu,ab} h^{a\mu} \right\} V^{\rho}(\bar{z})$$

or

$$V_{//}^{\mu}(\bar{z} + d\bar{z}) = V^{\mu}(\bar{z}) - d\bar{z}^{\nu} \Gamma^{\mu}_{\rho\nu} V^{\rho}(\bar{z}) \quad (7)$$

owing to (1) and the relation $A_{\mu,ab} = -A_{\mu,ba}$. (7) is nothing but the definition of the well-known parallel transport

of a vector $V^{\mu}(\bar{z})$. This result implies that the ordinary

derivative $[\partial V^{\mu}(x)/\partial x^{\nu}]_{x=0}$ of a vector operator in

the microscopic space $\omega(p)$ corresponds to the covariant

derivative of a C -number world vector $\delta_{\nu} V^{\mu}(\bar{z}) = \frac{\partial V^{\mu}}{\partial \bar{z}^{\nu}} +$

$+\Gamma^{\mu}_{\rho\nu}(\bar{z}) V^{\rho}(\bar{z})$ in the macroscopic world. Thus we

arrive at the conclusion that any Lorentz-covariant \mathcal{F} -number

equation in the microscopic space is translated to a general

covariant C -number equation in the macroscopic world by

taking the expectation value of the former.

The equation (2) or its equivalent

$$\nabla_{\mu} \Psi(\xi) \equiv \partial \Psi / \partial \xi^{\mu} - \frac{i}{2} A_{\mu,ab}(\xi) \cdot M^{ab} \Psi(\xi) = 0 \quad (8)$$

is covariant under the following two kinds of transformations

i) general ~~coordinate~~ transformation of coordinate

$$\xi^{\mu} \rightarrow \xi^{\mu'} = f^{\mu}(\xi)$$

$$A_{\mu,ab} \rightarrow A_{\mu',ab}(\xi') = \partial \xi^{\nu} / \partial \xi^{\mu} \cdot A_{\nu,ab}(\xi)$$

$$\Psi(\xi) \rightarrow \Psi'(\xi') = \Psi(\xi)$$

ii) generalized Lorentz transformation

$$h^{\mu}_{\nu}(\xi) \rightarrow h^{\mu'}_{\nu}(\xi) = h^{\mu}_{\nu}(\xi) + \epsilon^{\mu}_{\nu}(\xi), h^{\mu}_{\mu}(\xi)$$

$$A_{\mu,ab}(\xi) \rightarrow A_{\mu',ab}(\xi) = A_{\mu,ab} + \epsilon_a^k \cdot A_{\mu,kb} + \epsilon_b^k \cdot A_{\mu,ak} + \partial \epsilon_{ab} / \partial \xi^{\mu}$$

$$\Psi(\xi) \rightarrow \Psi'(\xi) = \left\{ 1 + \frac{i}{2} \epsilon_{ab}(\xi) \cdot M^{ab} \right\} \Psi(\xi)$$

$$\xi^{\mu} = \text{unchanged}$$

where $\epsilon_{ab} = -\epsilon_{ba}$.

In spite of these favourable properties, eq. (8) is not integrable as shown by

$$\left(\frac{\partial^2}{\partial \xi^\mu \partial \xi^\nu} - \frac{\partial^2}{\partial \xi^\nu \partial \xi^\mu} \right) \Psi(\xi)$$

$$= -\frac{i}{2} F_{\mu\nu, ab}(\xi) \cdot M^{ab} \Psi(\xi),$$

where

$$F_{\mu\nu, ab} = \partial_\mu A_{\nu, ab} - \partial_\nu A_{\mu, ab} - A_{\mu, ak} A_{\nu, kb} + A_{\nu, ak} A_{\mu, kb}.$$

We have employed the commutation relation (5) in deriving the above relation. It may be easily noticed that our $F_{\mu\nu, ab}$ is related with the Riemann-tensor \mathcal{R} in the following way⁽²⁾

$$F_{\mu\nu, ab} = -h_a^\rho \cdot h_b^\sigma \cdot \mathcal{R}_{\rho\sigma, \mu\nu}.$$

Therefore if the world is not flat, a state vector Ψ at ξ is not only a function of ξ but depends also upon the curve C along which Ψ was integrated.

Let us consider two curves C_1 and C_2 both issuing from a definite point P and intersecting each other at another point Q . Let a state vector $\Psi(P)$ be transferred following the equation (8) along the curves C_1 and C_2 respectively.

Then we have two state vectors at Q , i.e., $\Psi_1 = \Psi(Q, C_1)$ and $\Psi_2 = \Psi(Q, C_2)$, the difference of these is given by

$$\Delta\Psi(Q) \equiv \Psi(Q, C_1) - \Psi(Q, C_2)$$

$$= -\frac{i}{2} \iint_S d\sigma^{\mu\nu} F_{\mu\nu, ab} M^{ab} \Psi(P), \quad (9)$$

where S is a very narrow two-dimensional surface hemmed by the curves C_1 and C_2 . The right-hand side of (9) depends not only upon the curves C_1 and C_2 but also upon the surface S . This fact contradicts the left-hand side of (9) which shows Ψ depending only upon the curve. This contradiction, however, can be resolved approximately by virtue of Bianchi's identity when the gravitational field is so weak that the linear approximation is valid.

On the contrary when the gravitational field is not weak, it seems necessary to modify the equation of gravitational field in order to get rid of the above mentioned difficulty. In this connection it is very interesting to recall the fact that the phase of a state vector is undetermined. One is able to follow, in the present case, Dirac's ⁽³⁾ line of thought developed in his theory of magnetic single pole where the indeterminacy of phase was effectively employed. This kind of argument together with a suitable change of field equations may lead to a quantization of the source of A -field.

Reference

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Biography

Name: Ryoyu Utiyama

Date of Birth: August 28, 1916

Birth Place: Takajo-machi, Shizuoka, Japan

Education and Occupation

March 31, 1940: graduated from Dept. of Phys.
Osaka University, Osaka, Japan

Nov. 1940: Research Associate of the same department

Sept. 1942: Lecturer

March, 1945: Assistant Professor

March 1951: received Doctor Degree (old system)

Oct. 1955: Full Professor

Aug. 1954 - May 1956: visiting member of Institute in Princeton, N.J.,
U.S.A.

Sept. 1960 - Oct. 1961: visiting member of the Institute in Chapel Hill,
N. C., U.S.A.