LOCAL IRREGULARITIES IN A GÖDEL UNIVERSE, AND MACH'S PRINCIPLE

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Abstract

The effects of local irregularities on the equations of motion for a test-particle in a Gödel universe are considered by use of Einstein's field equations of gravitation. This investigation leads to a verification of the relationship, often referred to as Mach's principle, between the kinematical and gravitational properties of matter.

I'll put a girdle round about the earth
in forty minutes.[1]

Some fifteen years ago Kurt Gödel[2] discovered a new cosmological solution to Einstein's field equations of gravitation, in which there is uniform rotation of matter relative to the compass of inertia. This solution has certain undesirable physical characteristics, such as the non-existence of a one-parametric system of three-spaces everywhere orthogonal on the world-lines of matter and a uniform density of matter throughout an infinite cylinder. A consequence is that it is not possible to define uniquely a universal time coordinate. More complicated solutions containing non-uniform rotation of matter have been obtained subsequently.[3] However, the essential nature of rotation has remained highly elusory to relativists, and no satisfactory physical model incorporating rotation has yet been established. In this essay a means of obtaining a physically realizable interpretation of the Gödel universe is presented.

Perturbations have long provided our main means of interpreting Einstein's theory. By regarding the actual, curved, space-time as a first or second approximation to Galilean, or flat, space-time, all the usable information that the theory yields has been forthcoming. One of the major difficulties with cosmological solutions is that a mean density is used, and no account is taken of local anisotropies in the mass distribution. In a recent paper, Irvine[4] has considered the effects of such mass irregularities against the background of an expanding universe. A similar approach is used here for the Gödel universe. There are essentially two stages involved in the argument. Firstly, a local fluctuation is superimposed onto the Gödel metric, and the corresponding field equations and equations of motion for a test-particle are evaluated. Secondly, a locally inertial system of coordinates is used to obtain a particular observer's description of the universe in which he uses proper measurements of empirical physical quantities.

The Gödel metric may be written in the form:

$$(ds)^2 = \bar{g}_{ij} dx_i dx_j$$

$$= a^2 \left[(dx_0)^2 - (dx_1)^2 + \frac{e^{2x_1}}{2} (dx_2)^2 - (dx_3)^2 + 2e^{x_1} dx_0 dx_2 \right] .$$

Local perturbations γ_{ij} may be described by considering the metric:

$$g_{ij} = \bar{g}_{ij} + \gamma_{ij} .$$

These give rise to fluctuations in density and potential, which are related by the field equations:

$$R_{ik} = \frac{1}{2} R \, g_{ik} - \kappa \, T_{ik} ,$$

where the energy-momentum tensor T_{ik} is given by

$$T_{ik} = (\rho + p/c^2) \, u_i u_k - p/c^2 \, g_{ik} ,$$

$\kappa = 4\pi G/c^2$, and ρ and p are density and pressure respectively. In order to simplify the field equations it is necessary to linearize in terms of the components of the perturbed metric tensor. Furthermore, characteristic lengths, times, and velocities for the perturbation are taken to be quantities of first order in smallness. After some reduction, the field equations reduce to:

$$\phi^{\,|\lambda}_{\,|\lambda} = -4\pi G \tilde{\rho} ,$$

where $\phi = \frac{1}{2}(c^2/a^2)\gamma_{00}$, and the operator $^{|\lambda}_{|\lambda}$ denotes a generalization to three-dimensional curved space of the usual Laplacian. Thus, this is simply a generalized Poisson's equation.

We now define elements of proper length $d\ell$ and time $d\tau$, and a corresponding proper velocity $v = d\ell/d\tau$. These quantities may be used to make an infinitesimal coordinate transformation to the locally inertial coordinate system S in which they are measured. This will then yield a particular observer's description of the universe by means of his own local measurements. The resulting equation for the potential ϕ referred to S is:

$$\nabla'^2 \phi = 4\pi G \tilde{\rho},$$

where the prime indicates differentiation with respect to proper lengths. The density fluctuation $\tilde{\tilde{\rho}}$ is not necessarily small in magnitude when compared with the unperturbed density.

By an analogous procedure, the equations of motion for a test-particle in a neighbourhood of the given observer may be evaluated. Considerable simplification is obtained by the transformation to locally geodesic coordinates S and by using the proper velocity v of the mass irregularities relative to the local standard of rest. It is found that the equations satisfied by the perturbation are:

$$d\underline{v}/d\tau = -2\underline{\omega}_\wedge \underline{v} - \nabla'\phi - 1/(\rho + p/c^2)\nabla' p,$$

where $\underline{\omega} = (0, 0, -c/a\sqrt{2})$ is the angular velocity of matter relative to the compass of inertia in the Gödel universe. Terms on the right-hand side of this equation represent the effect of the Gödel universe on the behaviour of the perturbation, as observed in the observer's locally inertial reference system, S.

These equations of motion clearly reduce to a form analogous to that of Newton's equations of motion in an inertial frame, provided we write

$$\underline{V} = \underline{v} + \underline{\omega}_\wedge \underline{r}, \quad (\underline{r} = (\ell_1, \ell_2, 0)),$$

and define the operator:

$$D/D\tau = d/d\tau + \underline{\omega}_\wedge .$$

The equations of motion then take the form:

$$D\underline{V}/D\tau = -\nabla \cdot \tilde{\phi} - 1/(\rho + p/c^2) \quad \nabla \cdot p ,$$

where $\tilde{\phi} = \frac{1}{2} \omega^2 r^2 + \phi$, in a local reference frame S* rotating with angular velocity $\underline{\omega}$ with respect to the frame S. It appears to be possible to detect the rotation of distant matter by the appearance of a non-inertial term in the local equations of motion for a test-particle. In the locally inertial reference frame S, the departure from the Newtonian inertial equations of motion is caused by a Coriolis term, while in the frame S*, in which there is no relative motion of the matter of the Gödel universe, the Coriolis term is replaced by a centrifugal potential term. Moreover, the gravitational perturbation considered is arbitrary, to within satisfying requirements of smallness and the Poisson-like equation previously derived, relating it to the density perturbation. We have thus derived a relationship between the gravitational and kinematical effects induced by a uniformly rotating universe, with respect to either the locally inertial or the co-rotating reference frame of an observer at a point at which there is an arbitrary fluctuation in the density of matter.

Mach's principle has been expressed in a wide variety of forms, but it seems possible to categorize these statements into "weak" and "strong" versions. The "weak" version[5] postulates some relationship between the local inertial frame of the observer and the distant masses of the universe. However, the choice of a particular cosmological solution to Einstein's field equations determines uniquely the local compass of inertia, which may be defined as the tangent at the given point to the world-line of matter through that point. Equivalently, we may say that the locally inertial system of the given observer has been defined; in other words, the "weak" form of Mach's principle is trivially satisfied, once the cosmology is introduced explicitly.

On the other hand, the "strong" form of Mach's principle approximates to Mach's original suggestion[6], made in 1873. He emphasized that the empirically observable effects of the rotation of the earth relative to the distant masses of the universe should be indistinguishable from the rotation of the latter relative to the earth. Clearly, the local mass irregularities which we have considered have this property relative to the matter in the Gödel universe.

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