The Polarization of Radiation in General Relativity

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Introduction

Recent work in general relativity has clarified the concept of gravitational radiation. Of particular importance is the idea of null radiation, which is the gravitational analogue of null electromagnetic radiation, in which the electric and magnetic fields are equal in magnitude and at right angles to one another. This configuration of electromagnetic fields is well known in the flat space-time of special relativity. It has also been discussed in general relativity in connexion with plane and plane-fronted wave solutions of the Einstein-Maxwell equations. Nevertheless its role in general relativity is not clear. In particular it [illegible] fied" field theory, which are described in the next section [illegible] is the main aim of this [illegible] difficulties. Such a solution is a necessary preliminary to a full understanding of both [illegible] magnetic and gravitational radiation.

Already Unified Field Theory

The historical aim of unified field theory is to extend the basis of general relativity in such a way that the electromagnetic field is described by geometrical quantities. It was pointed out by Rainich and by Misner and Wheeler, that there is a sense in which this aim is already achieved in conventional general relativity. For the content of the Einstein-Maxwell charge-free equations can be expressed entirely in terms of equations for geometrical quantities. If these geometrical equations are satisfied, the electromagnetic field can be computed almost completely — only an unimportant constant (which can be added to the so-called "complexion" of the field) is left undetermined. It follows that the electromagnetic field is already described by geometrical quantities in the framework of general relativity.

Unfortunately for this program, it breaks down when the electromagnetic field is null. This breakdown has two aspects. In the first, stressed by Misner and Wheeler themselves and by Witten, one of the basic equations of the theory becomes empty. In the second, stressed by Penrose, the undetermined constant of the non-null case here becomes an undetermined function of position. This can be expressed physically by saying that the geometrical quantities do not determine the polarization of the null electromagnetic wave. In particular the polarization can change arbitrarily with position. Consider now two electromagnetic wave packets, destined later to collide, but initially separated by a patch of null field. Even if we fix the complexion of one wave packet, the geometry does not determine the complexion of the other, since the polarization can vary arbitrarily across the null patch. But when the wave packets later collide, the total electromagnetic field depends critically on the relative complexions. In other words, the geometry at a later time is not completely determined by the geometry at an earlier time so that the system is not completely describable by geometrical quantities

If we wish to avoid this difficulty we must find a geometrical way of specifying the polarization of a null wave. Now it is well known that there is a close connexion between the polarization of an electromagnetic wave and its spin angular momentum. We can therefore rephrase the problem as follows: is there a geometrical way of describing the spin of an electromagnetic wave? In order to answer this question we must first describe a theory proposed recently by the author and independently by Kibble.

Geometrical Theory of Spin

It is the basic tenet of this theory that the flux of material spin angular momentum is given geometrically by the skew part of the affine connexion of space-time. This skew part (called "torsion" by differential geometers) vanishes in Riemannian geometry, so we have here an extension of the geometrical basis of general relativity. There is no space to defend this extension here; rather we wish to generalize the theory so as to include electromagnetic spin as well as material spin, and to indicate how this eliminates the Penrose difficulty

This generalization is not quite straightforward since the formal definition of the electromagnetic spin flux is not gauge-invariant. In special relativistic quantum theory it was shown by Bargmann and Wigner that to obtain a gauge-invariant quantity one must take the completely skew part of the spin flux operator, or equivalently the vector operator which is its dual. The corresponding procedure in general relativistic non-quantum theory is to take a certain linear combination of the covariant derivatives of the electromagnetic field. This combination is also a vector which we shall call the polarization vector; its significance can be expressed in the following way.

An electromagnetic field defines at each point of space-time two orthogonal planes (or "blades"). If the field is non-null these two planes have no vectors in common. If it is null they share a null vector, which is the propagation vector of the wave. In either case what we have called the polarization vector measures how much the blades rotate as one moves from point to point of space-time; strictly speaking, therefore, we should call it the change-of-polarization vector.

In the non-null case it has been shown by Rosen and Bertotti that the polarization vector enters into one of the Rainich equations. This equation in fact asserts that the vector is the gradient of the complexion of the field. In the null case this Rainich equation is empty, but the polarization vector is still definable in terms of the electromagnetic field, and is still a gradient. However it is not determined by any quantities entering into strictly Riemannian geometry.

Geometrical Theory of Polarization

It is clear now what we have to do. We have to relate the torsion tensor to the polarization vector. First it is necessary to assume that the torsion tensor is completely determined by its contraction, the torsion vector. We then set this torsion vector equal to the polarization vector. The mathematical details of this procedure will be described elsewhere; here we can give only a brief outline of what is involved.

Our equations are now

(i) the original Rainich-Misner-Wheeler equations (one of which is empty for a null field)

(ii) the equality of torsion and polarization vectors

(iii) the equation showing that the torsion tensor is determined by the torsion vector

These equations must satisfy a consistency check, namely that the torsion vector be the gradient of a scalar (the complexion) in both the non-null and the null cases. Fortunately this result can be deduced from the geometrical equations (which can themselves be deduced from an action principle).

It follows that when the torsion vector is given we know the complexion of the field (up to an unimportant additive constant), throughout space-time, no matter whether the field is null or non-null. We thus no longer have the freedom needed to apply the Penrose argument. In fact the geometrical variables give a complete description of the physical situation.

Conclusions

Since the theory we propose is difficult to follow without its mathematical details, we should like to recall here its basic ideas. We have traced the breakdown of already unified field theory to the inadequacy of the Riemannian geometry on which it is based. Following the lead of an existing theory of the author and Kibble, we propose to extend the geometry by admitting the existence of a non-vanishing torsion vector. This vector can be set equal to the gradient of the complexion in the non-null case, and to the rate of change of polarization in the null case. The electromagnetic field is now completely determined by the geometry up to an unimportant arbitrary constant that can be added to the complexion.

One interesting point remains to be investigated. This is the nature of the solutions of the new equations in the presence of Wheeler-Misner wormholes. In particular it may turn out that the complexion can always be chosen so that there are no magnetic poles. If so, already unified field theory would at last have a new physical result to its credit.

Finally we should mention that these ideas may shed light on the nature of the spin and polarization of gravitational waves. There will be interesting differences, since the gravitational field possesses a super-energy tensor (Robinson, Bel, Sachs) rather than an energy tensor; so perhaps it possesses a superspin also. But, however it turns out, it appears that the polarization-spin is likely to be an important property of gravitational radiation.