A New Approach to Isolated Systems in General Relativity

Essay on the Theory of Gravitation

by

B. G. Schmidt

München

Summary. A new method to formulate asymptotic conditions for the gravitational field of isolated systems is presented. It is based on a boundary attached to spacetime, which is determined by the conformal structure intrinsically. The boundary is a generalisation of $\mathcal{I}$, defined by Penrose, and constructed using a certain bundle over spacetime.

Application to asymptotically simple spacetimes shows that the boundary contains not only $\mathcal{I}$ but further points, corresponding I^0, I^-, I^+ in Minkowski space.

Sufficient conditions for I^0 to be a point are given. In this case one gets the Poincaré group naturally as an asymptotic symmetry group.

Further issues, where this new boundary may be quite useful, are discussed.

A New Approach to Isolated Systems in General Relativity.

The idea of isolated systems is intimately related to a concept of a symptotic flatness. Bounded sources should determine a spacetime which becomes more and more like Minkowski space far away from the sources.

Essentially two notions of asymptotic flatness are used in General Relativity.

In the context of gravitational radiation Sachs and Bondi formulated asymptotic conditions along outgoing null hypersurfaces. These conditions were further developed and recast by Penrose into the definition of "future null infinity" called $\mathscr{I}$. Spacetimes which possess $\mathscr{I}$ are called "weakly asymptotically simple" and behave in a precisely defined sense like Minkowski space along null geodesics which terminate at .

For static and stationary solutions a quite different asymptotic flatness condition has proved to be useful. One demands that a spacelike hypersurface behaves more and more like a 3-plane in Minkowski space. Geroch developed this further in defining spacelike infinity also for non-stationary spacetimes using similar technics as Penrose.

Up to now no relation between the two approaches is known.

In a recent paper I developed a generalisation of the b-boundary construction applicable to the conformal and projective structure of spacetime. The conformal boundary one gets seems to be very useful to formulate asymptotic flatness conditions in a intrinsic and natural way.

First the construction of the conformal boundary will be described briefly, then some new results will be reported and finally some important problems are mentioned, which can be tackled from a new point of view.

A conformal structure can be defined as the reduction P of the frame bundle consisting of all frames which are orthonormal in any metric in the conformal class. The collection of connections, defined by the metrics in the conformal class, define a further reduction P^1 of the frame bundle of P. On the bundle P^1 there exists a natural parallelisation determined by the conformal structure, which is used to define a positive definite metric on P^1. The parallelisation is determined as follows: any connection of a metric in the conformal class defines a section in P^1. Sections passing through the same point with different tangent directions have different Ricci tensors. Under all subspaces of the tangent space one gets this way, there is a unique one determined by the condition that the Ricci tensor vanishes. This complement to the tangent space of the fibre defines the parallelisation.

Constructing the Cauchy completion one defines a boundary of P^1 and via the projection one gets a boundary of spacetime, intrinsically defined by its conformal structure. Boundary points can be characterised in the following way: Take a curve $x(\lambda)$ inextensible in V^4. Determine a connection in the conformal class whose Ricci tensor vanishes along $x(\lambda)$. (This is always possible). If the generalised affine length given by this connection is finite, then the curve defines a point in the boundary.

For Minkowski space the conformal boundary turns out to be $\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$ together with the three points I^-, I^0, I^+. Hence one gets precisely the boundary attached to Minkowski space by conformal imbedding into the Einstein universe.

The projective structure of spacetime defines a boundary $\partial_p V^4$ in a quite analogous way as described for the conformal strucure above.

For Minkowski space the boundary coincides with the one, one gets by the natural projective imbedding of Minkowski space into a 4-sphere. This agrees with the definition of "future projective infinity" defined by Eardley and Sachs recently. The interesting point is that timelike geodesics which in the conformal boundary all terminate at one point I^0, terminate in the projective case on a hypersurface.

Therefore one might conjecture that generally the projective boundary will be useful to describe the behaviour of matter in the distant future.

Constructing the conformal boundary $\partial_c V^4$ of a weakly asymptotically simple spacetime one finds that $\mathcal{J}$ is contained in $\partial_c V^4$. More interesting however is that any generator of $\mathcal{J}$ gets a future and past endpoint in $\partial_c V^4$! These sets of boundary points are denoted by I^-, $I^o(\mathscr{I}^-)$, $I^o(\mathscr{I}^+)$, I^+. It is, however, not true that the endpoints are identified always to form just three points as for Minkowski space. Therefore weakly asymptotically simple spacetimes can be naturally classified according to their structure of I^o.

The class which reflects most of the asymptotic properties of Minkowski space is the one in which I^o is one point. This is for example the case for the Schwarzschild solution.

In general I have so far only been able to find sufficient conditions for the structure of $\mathscr{I}^+$ which imply that $I^o(\mathscr{I}^+)$ is a point. The conditions are essentially that the new s function tends to zero there. This indicates a relation between the structure of $I^o(\mathscr{I}^+)$ and the amount of radiation produced by the source in the infinite past, which is physically quite plausible. Necessary and sufficient conditions for $I^o(\mathscr{I}^+)$ to be a point and their relation to the outgoing radiation field have to be found by further investigations.

Suppose that $I^o(\mathscr{I}^+)$ is a point. Then one can show that there is a uniquely defined action of the Poincaré group on $\mathscr{I}^+$!

In the case of Minkowski space one can find the Poincaré group in the Bondi Metzner Sachs group as the subgroup of those transformations which are regular at the point I^o. In the general case, as long as $I^o(\mathscr{I}^+)$ is a point, even a singular one, there remains sufficient regularity along $\mathcal{J}^+$ to single the Poincare group out of the BMS group. This result is of major importance, because it implies the possibility to define energy-momentum and angular momentum. The action of the Poincaré group on $\mathcal{J}^+$ defines a collection of canonical slices of $\mathcal{J}^+$ uniquely up to Poincaré transformations. Using the expressions of Tambarino and Winicour and the canonical slices one can define energy-momentum, angular momentum and calculate the change of these quantities in the radiation process. Because of the supertranslation freedom in the Bondi Metzner Sachs group it was up to now not possible to define angular momentum.

There is a further aspect under which these canonical slices might prove useful. In Newmans approach to equations of motions one major problem is that there is too much freedom in choosing slices on $\mathcal{J}^+$. Using the canonical slices one can resolve this difficulty[1).]

The canonical slices also define a preferred class of coordinate systems near $\mathcal{J}^+$ which are uniquely defined by the slicing of $\mathcal{J}^+$. These coordinate systems can be used to linearise Einsteins equation near $\mathcal{J}^+$. This way one gets a much smaller gauge group as usual.

---

1) This point was realised by Martin Walker.

These are the results obtained so far. Let us now turn to further problems which can be dealt with using the conformal boundary.

Cauchy data on a spacelike hypersurface determine uniquely a spacetime. Hence the data specify also the conformal boundary. Its a formidable task to find conditions on the data which imply the existence of $\mathcal{J}$ and a certain structure of I^o. A simpler question is to ask for conditions on the asymptotic behaviour of the data which guarantee the existence of a "piece of $\mathcal{J}$" and a certain structure of I^o. The conformal boundary will certainly be a useful tool in this context and hopefully its application will bring some insight into the relation between null and spacelike asymptotic flatness.

Related to this is the problem of incoming radiation on $\mathcal{J}^-$. For a truly isolated system the radiation field on $\mathcal{J}^-$ should vanish. It is, however, completely unclear whether non-stationary solution satisfying this condition exists at all!

The structure of I^o which relates in some sense $\mathcal{J}^+$ and $\mathcal{J}^-$ may give first indications. Somehow it seems puzzling that a change in the sign of the second fundamental form of the initial surface should shift the radiation field from $\mathcal{J}^+$ to $\mathcal{J}^-$.

The results obtained so far and the whole range of problems which can be reconsidered from a new point of view, show that the conformal boundary is a useful concept in General Relativity.

Hopefully it will lead to further insight into the structure of Einstein's theory of gravitation.

Acknowledgement: This essay would not have been written in this form without many discussions with Martin Walker.

B. G. Schmidt
Max-Planck-Institut für
Physik und Astrophysik
8 München 40
Föhringer Ring 6

Biographical Sketch

I was graduated in Hamburg where I got the Ph.D. 1968. Since 1964 I was doing research in P. Jordans "Seminar on General Relativity". On invitation of D. Sciama I spent the academic year 1968/69 at the Department of Applied Mathematics and Theoretical Physics in Cambridge/England. From 1970 - 1972 I was an assistent at the University in Hamburg, where I aquired the degree of habilitation 1971. Presently I am working at the Max-Planck-Institut für Physik und Astrophysik in Munich in the group of Prof. J. Ehlers.