Black Hole Physics and the Universalities of Superradiance and of Grey–body Radiation

M. Schiffer*
Instituto de Matemática, Estatística e Ciências da Computação
Departamento de Matemática Aplicada – Unicamp CP6065
13081-970 Campinas, SP, Brazil

Abstract

In this essay we compare the response of a black hole to incoming radiation to that of a system consisting of a hot source hidden behind a semi-transparent mirror, and they happen to agree. Then, we display a thermodynamical proof showing that this agreement is not incidental: it is a universal feature of an ideal grey body. As a by-product of this argument the universality of superradiance emerges: absorptive media in rotation instead of damping incoming radiation is responsible for its amplification in superradiant modes. Our main conclusion here is that the black hole response to incoming radiation and superandiance are not features that arise because black holes are "exceptional" systems but, on the contrary, because they are very "ordinary" in the sense that they fall into the category of ideal grey bodies.

One hundred years ago Kirchhoff's put forward his famous law stating that the ratio between emissivity and absorptivity of a grey body for each frequency interval independs of the system in question, it is a universal function of the temperature of the radiation in which it is immersed in. Just a few years later, Planck worried with the issue of equilibrium between matter and radiation showed that (in a modern language) this ratio is nothing but the mean occupation number of photons in the mode in question. From the (early) rules of quantum theory, Planck obtained the emission probability of a given number of quanta by a perfect black body in a given mode:

$$p_{\rm BB}(n) = (1 - e^{-\beta})e^{-\beta n} \tag{1}$$

where $\beta = \hbar\omega/kT$.

Although black bodies are but an idealization because most systems in nature absorb partially incoming radiation, no generalization of Planck's distribution for absorptive systems appeared since then, perhaps because of the

*e.mail:schiffer@ime.unicamp.br

widespread feeling that unlike black body radiation, the radiation emitted by a grey body is not submitted to any universal law.

Exactly twenty years ago Hawking [1] showed the remarkable result that a black hole is submitted to the same universal laws Planck put forward at the turn of the century, it emits (spontaneously) radiation as if it were an ordinary hot body of temperature $T_{\rm BH}$, proportional to its surface gravity, and of absorptivity Γ, the (mode dependent) transmission coefficient through the potential surrounding the black hole. Namely, this probability of spontaneous emission of n quanta is:

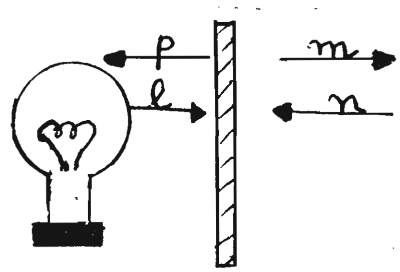
$$p(n) = \frac{e^x - 1}{e^x - 1 + \Gamma}\left[\frac{\Gamma}{e^x - 1 + \Gamma}\right]^n \tag{2}$$

where $x = \hbar\omega/kT_{\rm BH}$. In a latter development of the subject, Bekenstein and Meisels [2] and Panangaden and Wald [3] obtained the generalisation of emission probability for the case when the hole is impinged by n quanta:

$$\begin{aligned} p_{\rm BH}(m|n) &= \frac{(e^x-1)e^{xn}\Gamma^{m+n}}{(e^x-1+\Gamma)^{m+n+1}} \times \\ &\sum_{k=0}^{\min(m,n)} \frac{(-1)^k(m+n-k)!}{k!(n-k)!(m-k)!}\left[1 - 4\frac{1-\Gamma}{\Gamma^2}\sinh^2 x/2\right]^k \end{aligned} \tag{3}$$

The fact that the conditional probability of emission of quanta by a black hole looks so cumbersome is often credited to the complexity of this system. Very recently, it was reported [4] that the probability of detecting a given number of quanta in the radiation emerging from a medium consisting of a hot collection of harmonic oscillators reproduces exactly eq.(3). Intriguated with this coincidence, we scrutinezed a very simple system, a hot source hidden behind a semi-transparent mirror [5].

Assume a hot source located to the left of a semi-transparent mirror whose transmission and reflection coefficients are t and r, respectively ($|t|^2 + |r|^2 = 1$). Let l, n and m be the number of quanta emitted by the source, of incoming and emerging radiation at right hand side of the mirror, respectively.



The configuration of the hot source and mirror.

We associate to modes living on the right and left hand sides of the mirror the set of creation/annihilation operators a's and b's respectively. Accordingly, the initial state is:

$$|\overset{l}{\rightarrow},\overset{n}{\leftarrow}\rangle = \frac{{b^\dagger_\rightarrow}^l}{\sqrt{l!}}\frac{{a^\dagger_\leftarrow}^n}{\sqrt{n!}}|0\rangle. \tag{4}$$

Incoming and outgoing modes are related by the relations:

$$\begin{aligned} b^\dagger_\rightarrow &= r^* b^\dagger_\leftarrow + t^* a^\dagger_\rightarrow \\ a^\dagger_\leftarrow &= t\, b^\dagger_\leftarrow - r\, a^\dagger_\rightarrow \end{aligned} \tag{5}$$

Taking the scalar product with a state containt m right-outgoing quanta and defining the auxiliary quantities $x = -|r/t|^2$, the amplitude of detecting m quanta reads

$$\langle\overset{p}{\leftarrow},\overset{m}{\rightarrow}|\overset{l}{\rightarrow},\overset{n}{\leftarrow}\rangle = \sqrt{\frac{m!(l+n-m)!}{l!n!}}(-r)^{m-l}t^{n-m}|t|^{2l}x^{l-m}G(l) \tag{6}$$

where

$$G(l) = \sum_{k=m-l}^{\min(m,n)} x^k \binom{l}{m-k}\binom{n}{k} \tag{7}$$

In order to take into account the thermal nature of the source, the module squared of the above amplitude was weighted with the thermal factor:

$$p(m|n)_{\text{hot mirror}} = \sum_{l=0}^{\infty}(1-e^{-y})e^{-yl}|\langle\overset{p}{\leftarrow},\overset{m}{\rightarrow}|\overset{l}{\rightarrow},\overset{n}{\leftarrow}\rangle|^2 \tag{8}$$

After some algebra, these steps lead to the following conditional probability

$$\begin{aligned} p_{\text{hot mirror}}(m|n) &= \frac{(e^y-1)e^{ny}|t|^{2(m+n)}}{(1-r^2e^{-y})^{m+n+1}} \times \\ & \sum_{k=0}^{\min(m,n)} \frac{(-1)^k(m+n-k)!}{k!(n-k)!(m-k)!}\left[1-4\frac{|r|^2}{|t|^4}\sinh^2(y/2)\right]^k \end{aligned} \tag{9}$$

With the identification $|t|^2 \longrightarrow \Gamma$ this expression and eq.(3) coincide.

It is intriguing that the response of three widely different systems like a black hole, a collection of harmonic oscillators at fixed temperature and a hot source behind a semi-transparent mirror to incoming radiation do all agree. This fact cannot be fortuitous, it must have a deeper meaning and reflect some sort of universality. Universality of what? To answer this question, we must first realize these systems share in common. They are hot systems that partially reflect radiation, that is, they are *grey bodies*. Thus, we are facing here a strong evidence of the universality of grey body radiation. We shall hereby report our recent demonstration [6] of the universality of grey body radiation. To start out, we need a definition of a grey body.

The Definition

An ideal grey body is a system such that the following conditions are fulfilled:

- Absorbs in the mean only a fraction of the radiation incident on it in each frequency mode.
- It has a well defined temperature T.
- Unless $T = 0$, the radiation emerging from it is always described by a mixed–state density matrix.
- This density matrix factors into matrices for each frequency mode.

All the three systems mentioned above fit these conditions.

Evidently grey body radiance must be more complicated than black body radiance because the a's are not all equal to unity. Thus the formula analogous to formula (1) must refer to the conditional probability $p_a(m|n)$ for the emission, in a mode with absorptivity a, of m quanta, given that n are incident in the same mode (focusing attention on a single mode is justified by the third item in the Definition). After having defined what an ideal grey body is, we state a set of axioms its radiation ought to satisfy.

The axioms

Beside the trivial conditions of positivity of the $p_a(m|n)$ and normalization in the sense $\sum_m p_a(m|n) = 1$ the conditional probability of an ideal grey body satisfies:

- 1. $\sum_m m p_a(m|n) = b + (1-a)n$ and similarly with $a \to a'$.
- 2. $p_a(m|n)e^{-xn} = p_a(n|m)e^{-xm}$ and similarly with $a \to a'$.
- 3. $\sum_k p_a(m|k)p_{a'}(k|n) = p_{a''}(m|n)$

The first condition expresses the expectation that the grey–body returns outward, in the mean, a fraction $(1-a)$ of the number n of incident quanta, plus those it spontaneously emits, here quantified by the emission coefficient b, which may in principle be a function of T, a and some other parameters implicit in the $p_a(m|n)$'s. The first condition is evidently necessary for a grey–body to attain equilibrium with thermal radiation at the same temperature. However, it is not sufficient; from Einstein's work [7] we know that detailed balance must hold in addition. This is exactly what our second condition stands for: the probability for the absorption of n quanta by a grey body in a black–body enviroment with consequent emission of m quanta equals the probability for the absorption of m quanta with emission of n quanta.

The third condition claims that the effect of two ideal grey bodies at the same temperature T, but having different a's, which process incoming radiation in sequence, is equivalent to the effect of a third ideal grey body at temperature T, but having some other absorptivity. This is a reasonable expectation because, combined system satisfies all the three items of the Definition [6].

The Consequences

From these set of axioms it is possible to prove:

- Kirchhoff's law.
  The emissivity of a hot body equals its absorptivity:

$$b = \frac{a}{e^x - 1}. \tag{10}$$

- Planck's distribution for spontaneous emission

$$p(m|0) = \frac{e^x - 1}{e^x - 1 + a}\left[\frac{a}{e^x - 1 + a}\right]^m \tag{11}$$

- Law of composition of absorptivities.
  The effective absorptivity a'' of a system composed of two grey bodies of absorptivities a and a' at common temperature T is:

$$a'' = a + a' - aa' \tag{12}$$

  Note that since a and a' lie in the range (0,1), so does a''.

- Response to incoming thermal radiation.
  A grey body responds to incident thermally distributed radiation at inverse temperature α by emitting thermally distributed radiation but at different inverse temperature γ:

$$\frac{1}{e^\gamma - 1} = \frac{a}{e^x - 1} + \frac{1 - a}{e^\alpha - 1}. \tag{13}$$

- Universality of grey body radiation.
  As consequence of the last result, the convolution of the conditional probability with an exponential distribution function yields the generating function:

$$f(z) \equiv \frac{1 - e^{-\gamma(z)}}{1 - z} e^{-\gamma(z)m} \tag{14}$$

  where $z = e^{-\alpha}$. The conditional probability emerges through a process of repeated differentiation of this expression:

$$p_a(m|n) = \frac{1}{n!}\frac{d^n f(z)}{dz^n}\bigg|_{z=0}, \tag{15}$$

which turns out to reproduce exactly eqs.(3 and 9)[2].

This result is immediately generalizable to a rotating and charged grey body in entire analogy to the black hole case. It suffices to replace x in our second Axiom) by $x = (\hbar\omega - m\hbar\Omega - e\Phi)/T$ where m denotes the azimuthal quantum number of the mode, e the charge of a quantum, Ω the angular frequency of the body, and Φ its electrical potential with respect to infinity. Since the emissivity b of a grey body is always positive definite, the absorptivity must be negative for all modes such that $x < 0$ [see eq. (10)]. Thus for all superradiance modes ($x < 0$) incoming radiation is amplified instead of being absorbed. This phenomenum of wave amplification was known to exist in the framework of black hole physics even before the discovery of black hole radiation and was readly recognized as a mechanism to extract energy from rotating/charged black holes [8].

What we are witnessing here is a thermodynamical demonstration of the universality of superradiance for absorptive media in rotation. This result is very surprising because the absorption coefficient is a quantity that stems from the purely microscopical properties of the system. An example due to Zel'dovich [9] is very illustrative. He considered a scalar field Ψ satisfying the equation:

$$\Box\Psi + a\frac{\partial\Psi}{\partial t} = 0\,, \tag{16}$$

which phenomenologically describes a wave propagating through an absoprtive medium of absorptivity a. Now suppose that this medium is set into rotation with angular velocity Ω and let primed/unprimed coordinatates refer to the laboratory frame and to the frame in which the medium is at rest, respectively. Now, the field satisfies this same equation only in the coordinate in which the medium is at rest,

$$\Box'\Psi + a\frac{\partial\Psi}{\partial t'} = 0\,, \tag{17}$$

Transforming this equation to the laboratory frame, yields [9]

$$\Box\Psi + a\gamma\left(\frac{\partial}{\partial t} + \Omega\,\frac{\partial}{\partial\phi}\right)\Psi = 0 \tag{18}$$

where γ is the Lorenz factor $\sqrt{1-(v/c)^2}$ and ϕ is the azimutal angle in the laboratory frame. Consider now a wave with cylindrical symmetry propagating through this medium, $\Psi = f(r)e^{-i\omega t + im\phi}$. Inserting this form of Ψ into eq. (18) reveals that the effective absorption coefficient for the rotating medium medium is no longer a but

$$a \rightarrow a\gamma(1 - \frac{m\Omega}{\omega}), \tag{19}$$

meaning that for all modes such that $\omega < m\Omega$ the effective absorptivity reverses sign and the medium operates as an amplifier rather than an absorber.

Two remarks to finalize. The first is related to present attempt to black hole radiance in order to obtain a correlation between incoming and outgoing radiations and, consequently, solve one of the most intriguing problems of theoretical physics: the apparent conversion of pure states into mixed ones in the course of black hole evaporation [for a recent review see [10]]. What we learned here is that since black holes fall into the class of ideal grey bodies one has to be very carefull while changing the rules of quantum mechanics in order not to hazard all the beautiful coincidences we observe at the macroscopical level for ordinary systems. Secondly, a historical remark. It is surprising that the universality of grey body radiation and the discovery of superradiance for ordinary systems did not follow its the natural historial path as a natural generalization of the theory of black bodies. Some seventy years had still to pass until the theory of black hole radiance supplied the clue for these universalities.

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