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THE STRUCTURE AND STABILITY OF ROTATING MASSIVE STARS IN GENERAL RELATIVITY

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Abstract

The structure equations for a rotating massive star in the general theory of relativity are presented and solved. The dynamical equations are then considered and the fundamental radial mode of oscillation is calculated from a variational principle. This is found to be zero, implying transition from stability to instability when

$$\frac{2GM}{RC^2} = 1.3 \left(\frac{M_o}{M}\right)^{\frac{1}{2}} + 0.015 \frac{\Omega^2 R^3}{GM}$$

where Ω is the angular velocity and R the radius of a star of mass M. This implies that stars less than $2 \times 10^7 M_\odot$ can reach the temperatures required to burn hydrogen and that more massive stars live for some 2×10^4 years before becoming unstable.

The suggestion by Hoyle and Fowler[1] that stars with masses of $10^6 - 10^{10}\ M_\odot$ may provide the energy for radio sources, and the subsequent discovery of quasars has stimulated considerable interest in the structure of very massive stars.[2] Iben[3] using a binding energy argument showed that within the framework of general relativity a spherical massive star becomes unstable long before it has contracted to the stage at which nuclear reactions become important. A similar conclusion was obtained by Chandrasekhar[4] using a detailed stability analysis on the spherically symmetric relativistic equations and calculating the relaxation oscillations from a variational principle. Similar results have been obtained by Fowler[5] using a virial theorem approach.

It was shown by Roxburgh[6], using a simple virial theorem approach, that rotation had a considerable stabilizing effect, changing the radius at which instability occurred by a very large factor. In view of the large change produced by rotation it is desirable to have a proper stability analysis of rotating massive stars in general relativity, since in the Newtonian theory it is known that the virial theorem gives results that may be wrong by a factor of four.[7] This problem is considered here.

The general relativistic equations have been given by Misner and Sharp[9] and by Chandrasekhar[10] in a weak field approximation. This is sufficient for our purpose. If we confine our attention to slow rotation and integrate the equations over latitude, so leaving purely radial variables, the equations that govern the structure of the rotating massive star can be expressed as

$$\frac{2}{3}\,\Omega^2 r - \frac{1}{\rho}\frac{dP}{dr} - \frac{GM_r}{r^2}\left(1 + \frac{P}{\rho c^2} + \frac{2\ GM_r}{rc^2}\right) - \frac{4\pi G\ Pr}{c^2} = 0 \qquad (1)$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \qquad (2)$$

$$\rho = \rho_0 + \frac{U}{c^2} \qquad (3)$$

$$\frac{U}{P} = \frac{3(\gamma - 1) - \beta(3\gamma - 4)}{\gamma - 1} = 3 - 3\beta/2, \; \gamma = 5/3.$$

$$\frac{d \log P}{d \log T} = \frac{(32 - 24\beta - 3\beta^2)}{(8 - 6\beta)} \simeq 4 - \frac{3\beta^2}{8}, \; \beta << 1 \qquad (4)$$

$$P = P_g + P_R = P_g/\beta \qquad (5)$$

$$P_g = \frac{R}{\mu}\, {}_{o} T, \; P_R = \frac{a\, T^4}{3}$$

Equation 4 is the adiabatic condition which is used since the star is in convective equilibrium. [1,5].

With $\beta << 1$ these equations are the same as those of the polytrope $n = 3$, and the equations are readily integrated. β is then given by Eddington's [12] equation which approximates for massive stars to

$$\beta = 8.6 \left(\frac{M_0}{M} \right)^{\frac{1}{2}} \qquad (6)$$

By introducing suitable dimensionless variables equations (1) to (5) reduce to

$$\frac{d\sigma}{d\xi} = (1 + 3\,q\sigma) \left[\frac{1}{3}\,\alpha\,\xi - \frac{m}{\xi^2} \left(1 + \frac{q\sigma}{1+3q\sigma} + \frac{8\,mq}{\xi} \right) - q\xi\sigma^4 \right] \qquad (7)$$

$$\frac{dm}{d\xi} = \xi^2 \; \sigma^3 \,(1 + 3\,q\sigma) \qquad (8)$$

where

$$\alpha = \frac{\Omega^2}{2\pi G \rho_c}, \quad q = \frac{P_c}{\rho_c c^2}, \qquad (9)$$

the former measures the degree of rotation, the second the effects of general relativity. These equations are readily solved subject to the boundary conditions

$$\sigma = 1,\ m = O(\xi^3) \text{ at } \xi = 0 \qquad (10)$$

To calculate the stability against radial oscillations we introduce an adiabatic displacement $\Delta r = \eta\, e^{i\omega t}$ into general relativistic equations and linearise in the perturbations. After some elimination this gives an eigenvalue equation to determine ω^2 and η. It can readily be shown that the problem is self adjoint and a variational principle derived to determine ω^2. This gives

$$\Sigma^2 \left[\int_0^{\xi_1} B\,\sigma^3 (1 + 3q\sigma)\left(1 + \frac{4mq}{\xi}\right) \eta^2\, \xi^2\, d\xi \right]$$

$$= \int_0^{\xi_1} B\,\sigma^3 \left\{ \left(\frac{1 + 4mq}{\xi}\right) \left[\frac{\Gamma\sigma}{4} (\xi\eta' + 2\eta)^2 + \alpha (1 + 3q\sigma)\eta^2 \xi^2 \right. \right. \qquad (11)$$

$$\left. + 2\eta^2 \left(\sigma'\xi - (1 + 3q\sigma)\frac{m}{\xi} \right) \right] - 2q\,\Gamma\eta\,\frac{m}{\xi}\,(\eta'\xi + 2\eta)$$

$$\left. - \eta^2 \frac{mq}{\xi} \left(2\sigma + 20\frac{m}{\xi}\right) \right\} d\xi$$

where

$$B = \frac{e^{-4q\sigma}}{e^{-q}}$$

$$\Sigma^2 = \frac{\omega^2}{4\pi G\rho_c} \qquad (12)$$

$$\Gamma = 4/3 + \beta/6$$

To calculate Σ^2 we took a trial function

$$\eta = \xi + \lambda_1 \xi^2 + \lambda_2 \xi^3 \tag{13}$$

where λ_1 and λ_2 are parameters to be varied so as to obtain the minimum value of Σ^2. We initially set $\lambda_2 = 0$ and varied λ_1 so as to minimize Σ^2; with this value of λ_1, λ_2 was then varied to obtain an improved value of Σ^2. The accuracy of our solution was estimated by applying the principle to the non relativistic case, and comparing with the results of Cowling and Newing[7]. Our trial function gave considerably better results than the linear trial function.

With Γ given by equation (12) and β by equation (6) we evaluated Σ^2 for $M/M_\odot = 10^6, 10^7, 10^8, 10^9, 10^{10}$ for different values of q and α. Instability was found to set in when

$$q_i = 0.425 \left(\frac{2\,GM}{R_I\, c^2} \right) = 0.55 \left(\frac{M_0}{M} \right)^{1/2} + 0.56\, \alpha$$

Since the massive object is like a polytrope of index 3, the maximum possible value of α is $3.95 \times 10^{-3.13}$ corresponding to $\frac{\Omega^2 R^3}{GM} \sim 0.4$, since the ratio of central density to mean density in a polytrope $n = 3$ is approximately 60. With $M/M_\odot = 10^{10}$ and α at this maximum value, the radius at which instability sets in is decreased by a factor of 400. Even small values of α can have a considerable effect. For sufficiently large masses and maximum α, the star becomes unstable at a radius

$$R_I = 193\, R_g ,$$

where R_g is the Schwarzschild radius $2\,GM/c^2$.

This should be compared to the results of the simple virial theorem analysis of Roxburgh[6] which gave $R_I = 250\, R_g$. The detailed analysis therefore confirms the general conclusions of the virial theorem analysis. Using results established earlier[6] we find that a contracting rapidly rotating massive star can then reach the hydrogen burning temperature if its mass is less than 2×10^7 $M_\odot$. For more massive objects the star contracts drawing on its internal energy until it reaches the instability radius $R_I = 193\ R_g$ which takes some 1.6×10^4 years.

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