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Are All Static Black Hole Solutions Spherically Symmetric?

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Abstract

The static black hole solutions to the Einstein-Maxwell equations are all spherically symmetric, as are many of the recently discovered black hole solutions in theories of gravity coupled to other forms of matter. However, counterexamples demonstrating that static black holes need not be spherically symmetric exist in theories, such as the standard electroweak model, with electrically charged massive vector fields. In such theories, a magnetically charged Reissner-Nordström solution with sufficiently small horizon radius is unstable against the development of a nonzero vector field outside the horizon. General arguments show that, for generic values of the magnetic charge, this field cannot be spherically symmetric. Explicit construction of the solution shows that it in fact has no rotational symmetry at all.

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One of the many remarkable aspects of black holes is the high degree of symmetry of the classical black hole solutions. Both of the static solutions that were discovered in the early days of general relativity — the Schwarzschild and the Reissner-Nordström — are spherically symmetric. At one time, one might have thought that this simply reflected the fact that spherically symmetric solutions are easier to find. However, it was shown two decades ago [1] that these are in fact the only static electrovac black hole solutions. Does this result generalize to gravity coupled to other types of matter; i.e., are static black holes always spherically symmetric? The answer [2], as we will describe below, is no.

The restrictions on the possible electrovac black holes can be viewed as just one instance of the "no-hair" results that limit the possible structure of black holes in a number of matter theories. This suggests that in seeking solutions that depart from spherical symmetry one should look to theories that do not admit no-hair theorems; these tend to be [3] theories that possess static soliton solutions in the absence of gravity. An important class of such theories is the spontaneously broken gauge theories that possess nonsingular magnetic monopole solutions [4] in the absence of gravity; the simplest example is the SU(2) gauge theory with the symmetry broken to the U(1) of electromagnetism by a triplet Higgs field. The elementary particles of this theory include, in addition to the massless photon, a spin-one particle with mass m and electric charge e and an electrically neutral spinless particle.

This theory possesses spherically symmetric black hole solutions [5] with nontrivial matter fields outside the horizon; these can be constructed by numerical integration of the field equations. They carry magnetic charge $1/e$, and may be viewed as Schwarzschild-like black holes embedded in the center of 't Hooft-Polyakov monopoles. In fact, one does not need a detailed examination of the field equations to show that such solutions exist. An analysis [6] of the small fluctuations about the Reissner-Nordström solution reveals an instability leading to the development of massive vector meson fields just outside the horizon whenever the horizon radius is less than a critical value of order m^{-1}; the existence of this instability indicates that there must be a nearby static solution with hair.

The physical basis for this instability is easily understood. Charged particles with nonzero spin in general have magnetic moments. In a sufficiently strong magnetic field, the energetic cost of producing a cloud of such particles can be more than offset by the energy gained by aligning their magnetic moments so as to partially shield the magnetic field.

Hence, the essential features needed for these new black holes are captured by a theory that includes, in addition to the electromagnetic and gravitational fields, a massive charged vector field W_μ with a magnetic moment fixed by an arbitrary parameter g. This theory has the flat spacetime Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|D_\mu W_\nu - D_\nu W_\mu|^2 - m^2 W_\mu^* W^\mu \\ &- \frac{ieg}{4}F^{\mu\nu}\left(W_\mu^* W_\nu - W_\nu^* W_\mu\right) - \frac{\lambda e^2}{4}\left|W_\mu^* W_\nu - W_\nu^* W_\mu\right|^2\end{aligned} \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength and $D_\mu = \partial_\mu - ieA_\mu$ denotes the electromagnetic gauge covariant derivative. For $g = 2$ and $\lambda = 1$ this theory is essentially the spontaneously broken gauge theory discussed above, but with the terms involving the scalar field omitted. By setting $g = 2$ and $\lambda = 1/\sin^2\theta_W$, we obtain instead a portion of the standard electroweak theory.

The electromagnetic vector potential must have a singularity on any surface enclosing a magnetic charge. In order that the resulting "Dirac string" singularity not be physically detectable by charged particles, the magnetic charge must be equal to q/e, with q either an integer or a half-integer. The black holes with $q = 1$ are just the spherically symmetric solutions described above.

Let us now consider magnetically charged black holes with $q \neq 1$ in the context of this theory. For any allowed value of q there is a Reissner-Nordström solution for which the W field vanishes identically. Analysis [7] of the small fluctuations about this solution shows an instability similar to that found in the $q = 1$ case. Hence, we are once again led to the existence of new black hole solutions with hair. What is new here is that these solutions cannot be spherically symmetric. To understand this, note that any static field can be expanded as a sum of generalized spherical harmonics multiplied by functions of r. These spherical harmonics are the eigenfunctions of the quantum mechanical angular momentum operator; a configuration is spherically symmetric if its expansion contains only harmonics with zero angular momentum. The harmonics used in the expansion must be must be appropriate to the type of field. For an electrically charged vector field in the presence of a magnetic monopole with magnetic charge q/e, one needs monopole vector harmonics [8]. These are the eigenfunctions of the total angular momentum operator for a spin-one particle with electrical charge e moving in the presence of the monopole. In addition to the usual orbital angular momentum $\mathbf{r} \times m\mathbf{v}$ and the spin angular momentum, there is an anomalous angular momentum of magnitude $q\hbar$ oriented along the line from the electric charge to the monopole. Because this anomalous contribution is perpendicular to the orbital angular momentum, it is impossible to have vanishing total angular momentum if the spin angular momentum is smaller than the anomalous component. Hence, if $q > 1$ there is no monopole vector harmonic with zero angular momentum.

Because these new solutions are not spherically symmetric, one would expect them to be quite difficult to obtain. However, a perturbative approach is available for the case where the Reissner-Nordström solution is just barely unstable. In this situation, one would expect the unstable modes to give an indication of the nature of the nearby stable solution. This suggests that we begin by identifying the terms in the energy for static configurations that are quadratic in the perturbations about the Reissner-Nordström solution. This gives a quadratic form that has a single negative eigenvalue $-\beta^2 m^2$, where β tends to zero as the horizon radius approaches the critical value for instability. Because the corresponding modes, which involve only W_μ, cannot be spherically symmetric, they must form a degenerate multiplet corresponding to an irreducible representation of the rotation group; let us label these ψ_μ^M.

We now write the vector field as a linear combination of these functions plus a remainder that is of higher order in β:

$$W_\mu = \sum_M k_M \psi_\mu^M + \tilde{W}_\mu . \tag{2}$$

Next, the field equations can be used to determine the leading order deviations of the electromagnetic field and of the metric from their Reissner-Nordström values in terms of W_μ. The resulting expressions, together with Eq. (2), must then be substituted back into the energy. The dominant terms in the energy then become a fourth order polynomial in the k_M, whose minimum determines the values of the k_M, and thus the leading approximation to the solution.

This program is particularly simple to carry out if $g > 0$ and $q \geq 1$. In this case, the ψ_μ^M form a multiplet of $2q-1$ functions of the form

$$\psi_t = 0, \qquad\qquad \psi_j^M = f(r) C_j^{q-1,M}(\theta,\phi) \tag{3}$$

where the $C_j^{q-1,M}(\theta,\phi)$ are vector spherical harmonics with total angular momentum quantum number $J = q-1$ and $f(r)$ is a function that is nonzero on the horizon and falls exponentially with distance for $r > m^{-1}$. Hence, the vector field outside the horizon is of the form

$$W_\mu = f(r)\Phi_\mu(\theta,\phi)\,. \tag{4}$$

Using the explicit forms for the monopole harmonics, one can show that $\Phi_\mu^* \Phi^\mu$ has $2(q-1)$ zeros on the unit sphere. This fact by itself makes it quite clear that spherical symmetry is impossible for $q \neq 1$. One could conceivably have axial symmetry, with the zeros either all coinciding or else lying at two antipodal points. However, detailed study of the solutions shows that this is not what happens. Instead, the zeros tend to be distributed as evenly as possible over the unit two-sphere, leading to a configuration without any rotational symmetry. This vector field configuration induces higher multipole components in the electromagnetic field and the metric, with multipoles of order up to $2(q-1)$. At large distances from the black hole these fall as appropriate powers of r.

At one time the known static black hole solutions were all algebraically simple and highly symmetric. In recent years we have learned that a number of types of matter allow black holes that are considerably more complex in that they have nontrivial fields outside the horizon. The work described here shows that the inclusion of magnetic charge gives rise to objects with even more structure — black holes that have no rotational symmetry at all.

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