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Testing Einstein's Inhomogeneous Gravitational Field Equations

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It is shown that all the historical tests of Einstein's gravitational theory test only his homogeneous field equations $R_{uv} = 0$.

We point out in this paper that celestial "Eotvos" experiments which measure the gravitational to inertial mass ratio of massive bodies like the Sun or planets, will be sensitive to features of Einstein's inhomogeneous field equations $R_{uv} = k(T_{uv} - Tg_{uv}/2)$.

Testing Einstein's Inhomogeneous Gravitational Field Equations

I. The Homogeneous Gravitational Field Equations

All of the past tests of Einstein's gravitational theory have tested his homogeneous field equations

$$R_{uv} = 0 \qquad 1.$$

R_{uv} is the Ricci curvature tensor of the space-time Riemannian geometry. Space-time is assumed to possess a Riemannian metric g_{uv} which yields the invariant line element

$$ds^2 = \sum_{\mu,\nu=1}^{4} g_{\mu\nu}\, dx^{\mu} dx^{\nu} \qquad 2.$$

In the vicinity of matter the g_{uv} deviate from their special relativity values, being obtained from the field equations (1). These field equations have the properties;

1. They are linear in second derivatives of the fields g_{uv}.
2. But they are non-linear in the first derivatives of the fields.

The static spherically symmetric solution to (1) is the well known Schwarzschild metric which is exhibited below to sufficient accuracy[1].

$$g_{44} = 1 - 2\left(\frac{Gm}{c^2 r}\right) + 2\beta\left(\frac{Gm}{c^2 r}\right)^2 + \cdots \qquad 3a.$$

$$g_{11} = g_{22} = g_{33} = -1 - 2\gamma\left(\frac{Gm}{c^2 r}\right) + \cdots \qquad 3b.$$

$$g_{\mu\neq\nu} = 0 \qquad 3c.$$

($\gamma = \beta = 1$ in Einstein's theory and are used in (3a,b,c) only as labels to facilitate future discussion in this paper.) Gm/c^2 is the integration constant of the homogeneous field equations, if G is identified with Newton's gravitational constant, c with the speed of light, then m is the Newtonian gravitational mass of a source. Note that (3a) reveals the nonlinearity of the homogeneous field equations. The above metric solution (3a,b,c) yields the historical tests of Einstein's theory;

1. The red shift of spectral lines results from the linear metric term of (3a).
2. The deflection of light and alteration of the transit time of light in a gravitational field results from a combination of the linear metric terms of (3a) and (3c).
3. The 43" per century advance of Mercury's perihelion position is determined by all metric terms shown in (3a,b,c), and is therefore the sole test of the non-linear aspects of Einstein's homogeneous field equations.

The linear part of the static field (3a,b,c) may be Lorentz transformed (LT) to yield new gravitational fields produced by sources moving at velocity $\overline{V}$.

$$g_{44}(LT) = -4\alpha''\left(\frac{Gm}{c^2r}\right)\frac{V^2}{c^2} \qquad \text{4a}$$

$$g_{4(1,2,3)} = 4\Delta\left(\frac{Gm}{c^2r}\right)\frac{V_{(1,2,3)}}{c} \qquad \text{4b}$$

($\Delta = \alpha''' = \frac{1+\gamma}{2} = 1$ in Einstein's theory and are only labels for use later)

The field (4b) when used for a rotating body like the Earth yields a net gravitational field which will cause a gyroscope in orbit around the Earth to have a precession of its spin axis which is proportional to the Earth rotation frequency. An experiment to test for this field (4b) is under development at Stanford University.

A second type of correction to the static fields (3a,b,c which must be made for moving sources are retardation (R) corrections, which account for the fact that the field equations (1) are actually wave equations which predict gravitational fields to propagate at the speed of light. The retarded fields are

$$g_{44}(R) = -\chi\left(\frac{Gm}{c^2r}\right)\frac{\vec{r}\cdot\vec{a}}{c^2} + \alpha'''\left(\frac{Gm}{c^2r}\right)\frac{(\vec{r}\cdot\vec{V})^2}{r^2c^2} \qquad 5.$$

($\chi = \alpha''' = 1$ in Einstein's theory) Retardation corrections to a gravitational theory are intimately related to the prediction of gravitational radiation which has not yet been experimentally detected.

So it is seen that all past or present gravitational experiments only measure the homogeneous field equation metric terms of Einstein's gravitational theory.

II. The Inhomogeneous Gravitational Field Equations

Recently this author has been calculating the rate at which a massive body is predicted to fall in gravitational theory. In order to solve this problem all the fields enumerated in (3,4,5) were required as well as an additional non-linear gravitational field which can be obtained only from the inhomogeneous gravitational field equations of Einstein

$$R_{uv} = k\,(T_{uv} - T\,g_{uv}/2) \qquad 6.$$

T_{uv} is the stress-energy tensor of matter sources, T is the trace of T_{uv}, $T = T^{uv}g_{uv}$. In solving (6) for several sources one gets a non-linear field contribution [2.,3.]

$$g_{44}(NL') = 2\alpha' \sum_{i \neq j} \frac{G^2 m_i m_j}{c^4 r_{ij}} \left(\frac{1}{|\bar{r}-\bar{r}_i|} + \frac{1}{|r-r_j|} \right) \qquad 7.$$

($\alpha' = 1$ in Einstein's theory). The field (7) can not be obtained from the homogeneous field equations (1).

In terms of all the fields produced by sources described above, the calculation of the gravitational to inertial mass ratio of an equilibrium sphere of many mass points, m_i, (e.g. a gas sphere in thermal-gravitational equilibrium) gives the result [3.,4.]

$$\frac{M_g}{M_i} = 1 + \left[\frac{\sum_{i\neq j} \frac{G m_i m_j}{r_{ij}}}{\sum_i m_i c^2}\right]\left[\frac{8\Delta - 4\beta - 3\gamma - \chi}{2} + \frac{2\beta + \chi - \alpha' - 2}{6}\right] \qquad 8.$$

$$= 1 \quad (\text{in Einstein's theory})$$

(Incidently, (8) predicts $m_g/m_i \neq 1$ in the Brans-Dicke gravitational theory).

Note the presence of α' in (8). A measurement of the rate that massive celestial bodies fall in external gravitational fields will therefore be the first experimental test of the inhomogeneous as well as homogeneous field equations of Einstein. Also it is significant that m_g/m_i is effected by the Δ and χ gravitational field terms which have not been experimentally detected yet either.

Elsewhere this author has discussed several experiments which can measure the m_g/m_i ratio of celestial bodies.[5.,6.]

1. If m_g/m_i of the Sun differs from 1 by order 10^{-5} the stable libration points of LaGrange for the Sun-Earth orbiting system move toward the Earth by about 500 kilometers. Rader ranging to a satellite placed near the stable points offers an experimental test of the Sun's m_g/m_i therefore.

2. If m_g/m_i of the Earth differs from 1 by order 10^{-9} then the moon orbit is polarized (displaced) toward the Sun by an amount of about 25 meters. This will be detectable in laser lunar ranging experiments.

The above experiments have the intrinsic value of being celestial "Eotvos" experiments. However we have here pointed out that such measurements are endowed with the additional significance of being the first proposed gravitation experiments which will measure consequences of Einstein's inhomogeneous gravitational field equations.

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