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ON "NEGATIVE" MASS IN THE THEORY OF GRAVITATION

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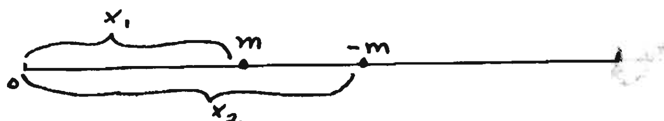
The purpose of this brief note is a critical survey of the idea of "negative" mass in the theory of gravitation. In reading Richard Ferrell's prize winning essay "The possibility of New Gravitational Effects", I found a basic confusion as to how such a "negative" mass would behave, and since subsequent discussions have convinced me that this confusion is widespread, I have taken this opportunity to attempt to clarify the issue.

The role which "mass" plays in the general theory of relativity is quite different from the "mass" of ordinary Newtonian mechanics and gravitation. It will be remembered that the true starting point of general relativity was noticing the equivalence of "inertial" mass (that is, the mass occurring in Newton's law of motion F = ma) and "Gravitational" mass (that is, the mass occurring in Newton's law of gravitation $F = -Gmm'/r^2$). This fact is, so to speak, built into the general theory of relativity. The result is that in discussing the motion of a mass point in general relativity the mass of the particle never appears. In the Newtonian case, on the other hand, the mass of the particle appeared on both sides of the equations, and then cancelled out due to the apparent fortuitous equality of inertial and gravitational masses. One can then ask: In view of this, how does mass enter into the general theory of relativity at all? The answer is that it enters in when we wish to discuss the field produced by an isolated material point. This solution of the partial differential equations of the gravitational field was first obtained approximately by Einstein and later exactly by Schwarzchild. This solution contains two arbitrary constants of integration. One is usually eliminated by requiring the gravitational field to be zero at infinity. The other is obtained by the requirement that sufficiently far away (i. e. when the gravitational field is weak enough) we must once more have Newton's law of gravitation. This enables us to determine the constant of integration in terms of the mass of the particle giving use to the field. Equivalently we may say that the mass of the particle is determined by the constant of integration. Now this is very striking, because this constant of integration is in principle completely arbitrary, and therefore the resulting mass is completely arbitrary; any positive or negative number whatsoever. That is, general relativity no more than Newtonian gravitational theory is capable of explaining the fact that all observed masses are positive. In other words, general relativity does not prohibit the existence of negative masses, even though such masses have never been observed. Up till now we agree with the statements of Ferrell; it is in the interpretation of what this negative mass means that we find a different result. We assert the following: A positive mass does not necessarily repel a mass which is, in the sense of general relativity, negative. The reason for this is as follows: The principle of equivalence of general relativity requires that the inertial mass equals the gravitational mass. If we take a particle with negative gravitational mass, then we must also take its inertial mass negative. A careful reading of Ferrell's paper shows that he defines negative mass as gravitational mass and inertial mass equal and opposite, a definition which is impossible according to the general theory of relativity. As an example, let us study in some detail the actual m motion of two particles, one of which has a mass m and the other a mass −m. Let them start from rest, and let their positions along the line joining them be given by x_1 and x_2 respectively.



Now we know that if we are not interested in extremely small corrections we may use Newton's laws to describe the motion. These are for our case

$$(m)\frac{d^2x_1}{dt^2} = \frac{Gm(-m)}{|x_1 - x_2|^2}$$

$$(-m)\frac{d^2x_2}{dt^2} = -\frac{Gm(-m)}{|x_1 - x_2|^2}$$

where G is the gravitational constant. Simplifying we obtain

$$\frac{d^2x_1}{dt^2} = -\frac{Gm}{|x_1 - x_2|^2}$$

$$\frac{d^2x_2}{dt^2} = -\frac{Gm}{|x_1 - x_2|^2}$$

Now if we subtract the first of these equations from the second we obtain:

$$\frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2} = 0$$

This equation may be integrated at once, and yields

$$x_2 - x_1 = a + bt$$

where a and b are constants of integration to be fixed by the initial positions and velocities of the two particles. Since we assumed the initial velocities to be zero we get b = o, and a will represent the initial distance the particles are apart. We have $x - x = a$. That is, <u>the distance between the particles remains constant</u>. <u>There is no trace of repulsion</u>. Now this result may be substituted back in our original equations. If we do this we obtain:

$$\frac{d^2x_1}{dt^2} = -\frac{Gm}{a^2}$$

$$\frac{d^2x_2}{dt^2} = -\frac{Gm}{a^2}$$

Therefore
$$x_1 = -\frac{Gm}{2a^2}t^2 + x_1^0$$

$$x_2 = -\frac{Gm}{2a^2}t^2 + x_2^0$$

We can describe the motion as follows: the particles remain at a fixed distance from each other, but the whole system accelerates uniformly to the left with the constant acceleration $\frac{Gm}{a^2}$.

Thus we see that for two masses starting from rest, one negative and the other positive, there is no repulsion at all between them but only the rather queer motion described above. The results of Ferrell would be obtained if one used the equations of motion*

$$(m)\frac{d^2x_1}{dt^2} = \frac{Gm(-m)}{|x_1 - x_2|^2}$$

$$(+m)\frac{d^2x_2}{dt^2} = \frac{Gm(-m)}{|x_1 - x_2|^2}$$

which are equivalent to

$$\frac{d^2x_1}{dt^2} = -\frac{Gm}{|x_1 - x_2|^2}$$

$$\frac{d^2x_2}{dt^2} = +\frac{Gm}{|x_1 - x_2|^2}$$

Adding we see that
$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = 0$$

or $x_1 + x_2 =$ constant. This means that the center of gravity stays fixed, as described by Ferrell. Similarly to the treatment given for the correct equations, one can easily show that these equations do give a repulsion between the particles. As pointed out above, however, these equations are in contradiction with the most basic principle of general relativity, the principle of equivalence.

In view of this result, we can ask if we have to abandon hope for the possibility of a gravitational shield, even if a nagative mass should be discovered in nature? The answer is no, if the negative mass is larger than the mass we wish to repel. To see this, let us go back to our old example, but now take the masses different. Let the positive mass be m as before the negative mass $-M$. The resulting equations of motion are

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*It is perhaps not without interest to note that these same equations would be obtained for the interaction of two negative masses. Thus two negative masses <u>would</u> repel each other in general relativity.

$$(m)\frac{d^2x_1}{dt^2} = +\frac{Gm(-M)}{|x_1 - x_2|^2} \quad \text{or} \quad \frac{d^2x_1}{dt^2} = -\frac{GM}{|x_1 - x_2|^2}$$

$$(-M)\frac{d^2x_2}{dt^2} = -\frac{Gm(-M)}{|x_1 - x_2|^2} \quad \text{or} \quad \frac{d^2x_2}{dt^2} = -\frac{Gm}{|x_1 - x_2|^2}$$

If we subtract the first of these equations from the second we obtain

$$\frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2} = \frac{G(M-m)}{(x_1 - x_2)^2}$$

or

$$\frac{d^2(x_2 - x_1)}{dt^2} = \frac{G(M-m)}{(x_1 - x_2)^2}$$

Now $x_2 - x_1$ is just the distance between particles. Call it r. Then the differential equation for r is

$$\frac{d^2r}{dt^2} = \frac{G(M-m)}{r^2}$$

This is a very well known equation, and we shall not describe its integration here. The result of one integration is

$$\left(\frac{dr}{dt}\right)^2 = -\frac{2G(M-m)}{r} + c$$

where c is a constant. If the particles are initially at rest and separated by a distance R, then

$$0 = -\frac{2G(M-m)}{R} + c \qquad c = \frac{2G(M-m)}{R}$$

$$\left(\frac{dr}{dt}\right)^2 = 2G(M-m)\left[\frac{1}{R} - \frac{1}{r}\right]$$

Since $\left(\frac{dr}{dt}\right)^2$ is always greater than zero, the right hand side of this equation must be greater than zero. We have three cases to consider.
(1) M greater than m. Then r must be greater than R and we see that we have repulsion since the distance tends to increase.
(2) M less than m. Then r must be less than R, and we see that there is attraction. Actually, a more careful investigation shows that the attraction lasts until the particles pass through one another (if they can), and then it is replaced by repulsion.
(3) M equal m. This is the case discussed previously, when there is neither attraction nor repulsion.

Thus we see that a large negative mass could act as a neutralizer of gravitation, but the size of the negative mass necessary depends on the mass of the object one wants to free from its gravitational bonds.