The Emergence of an Effective two - dimensional Quantum Description from the study of Critical Phenomena in Black Holes

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Abstract:

We study the occurrence of critical phenomena in four - dimensional, rotating and charged black holes, derive the critical exponents and show that they fulfill the scaling laws. Correlation functions critical exponents and Renormalization Group considerations assign an effective (spatial) dimension, $d = 2$, to the system. The two - dimensional Gaussian approximation to critical systems is shown to reproduce all the black hole's critical exponents. Higher order corrections (which are always relevant) are discussed. Identifying the two - dimensional surface with the event horizon and noting that generalization of scaling leads to conformal invariance and then to string theory, we arrive to 't Hooft's string interpretation of black holes. From this, a model for dealing with a coarse grained black hole quantization is proposed. We also give simple arguments that lead to a rough quantization of the black hole mass in units of the Planck mass, i. e. $M \simeq \frac{1}{\sqrt{2}} M_{pl}\sqrt{l}$ with a l positive integer and then, from this result, to the proportionality between quantum entropy and area.

Black hole dynamics can be described in terms of a set of laws that are the exact analogous of the ordinary four laws of thermodynamics[1]. This analogy became exact after Hawking's discovery[2] that, due to quantum effects, black holes are not black but radiate like a black body at a temperature proportional to its surface gravity. On the other hand, scaling of critical phenomena is a property that applies to a great variety of thermodynamical systems and have been extensively verified experimentally[3]. The Renormalization Group approach[4] gave a sound mathematical foundation to the concept of universality of the critical exponents. We show here that critical phenomena and scaling behaviour does also take place in black holes, these considered as a thermodynamical system (see also Ref. [5]).

Let us suppose that a rotating charged black hole is held in equilibrium with a surrounding heat bath at some temperature T. If we consider a small, reversible transfer of energy between the hole and its environment in such a way that the angular momentum J and charge Q remain unchanged, the full thermal capacity corresponding to this energy transfer can be computed by eliminating the black hole total mass, M, in the equations for the temperature and the area of the black hole, and differentiate the entropy, S, keeping J and Q constant[6],

$$C_{J,Q} = T\frac{\partial S}{\partial T}\bigg|_{J,Q} = \frac{MTS^3}{\pi J^2 + \frac{\pi}{4}Q^4 - T^2S^3} \ . \tag{1}$$

This heat capacity goes from negative values for a Schwarzschild black hole, to positive values for a nearly extreme Kerr - Newman black hole. Thus, $C_{J,Q}$ has changed sign at some J and Q in between. In fact, the heat capacity passes from negative to positive values through an infinite discontinuity.

Eliminating S and T in Eq. (1) by use of the expressions for the temperature and entropy of a black hole, the infinite discontinuity in $C_{J,Q}$ takes place at $j^2 + 6j + 4q = 3$ where $j = 8\pi\frac{J_c^2}{M^4}$ and $q = 8\pi\frac{Q_c^2}{M^2}$.

From eq(1) we can obtain $C_{JQ} \sim (T - T_c)^{-1}$, where the critical temperature is given by $T_c^{JQ} = \left\{2\pi M[3 + \sqrt{3-q}]\right\}^{-1}$, and q is given by the critical curve above. We find then the first two critical exponents (that measure the power at which the heat capacity diverges as we approach the critical curve at (J, Q) or T fixed), directly by inspection of C_{JQ}:

$$\alpha = 1 \quad , \quad \varphi = 1 \ . \tag{2}$$

Analogously, by noting that the isothermal compressibilities[7] $K_{T,Q}^{-1}$ and $K_{T,J}^{-1}$ diverge as $C_{J,Q}$ on the same singular segment as given above, we find the other two corresponding critical exponents

$$\gamma = 1 \quad , \quad 1 - \delta^{-1} = 1 \ . \tag{3}$$

To obtain the remaining critical exponents corresponding to the equation of state and the entropy, we choose a path either along a critical isotherm or at constant angular momentum $J = J_c$ or constant charge $Q = Q_c$. However, in this case the black hole equations of state just reproduce the critical curves and we can formally assign a zero power to the corresponding critical exponents:

$$\begin{gathered} \beta \to 0 \quad , \quad \delta^{-1} \to 0 \ , \\ 1 - \alpha = 0 \quad , \quad \psi \to 0 \ . \end{gathered} \tag{4}$$

One can easily verify that the set of critical values given by Eqs. (2) - (4) satisfy the scaling laws, e.g.

$$\begin{gathered} \alpha + 2\beta + \gamma = 2 \quad , \quad \alpha + \beta(\delta + 1) = 2 \ , \\ \gamma(\delta + 1) = (2 - \alpha)(\delta - 1) \quad , \quad \gamma = \beta(\delta - 1) \ , \\ (2 - \alpha)(\delta\psi - 1) + 1 = (1 - \alpha)\delta \quad , \quad \varphi + 2\psi - \delta^{-1} = 1 \ . \end{gathered} \tag{5}$$

Other five heat capacities can be computed, of which $C_{\Omega,Q}$ and $C_{J,\Phi}$ exhibit also a singular behavior[7]. Heat capacities and isothermal compressibilities at fixed (Ω, Q) and

(J, Φ) give exactly all the same critical exponents as in the previous case where we held (J, Q) constant. This result can in fact be understood as a realization of the *Universality hypothesis*[5].

The critical curves for the three cases studied are all different, but the critical exponents, according to the above mentioned hypothesis, are the same within each universality class[8]. We observe, in addition, that the equality between the primed ($T \to T_c^-$) and unprimed ($T \to T_c^+$) critical exponents (another characteristic of scaling) is also verified in each one of the three transitions studied.

Another typical feature of critical phenomena is that there is not such thing as a latent heat. This is also verified here since the internal energy of the black hole, e.g. M, remains continuous through the transition.

Not only relations among critical exponents of thermodynamic functions can be obtained, but also relations concerning exponents of correlation functions. Away but not far from the critical region, one can write the static two-point (at distance $|\vec{r}|$) connected correlations as $G_c^{(2)}(r) \sim r^{2-d-\eta} \exp\{-r/\xi\}$, for r large. Here d is the spatial dimensionality of the system, η is a further critical exponent and ξ is the correlation length. As one approaches the critical curve ξ diverges as $\xi \sim |T - T_c|^{-\nu}$ (here ν is another critical exponent). In thermal equilibrium, the correlation function of a scalar field ϕ (playing the role of the order parameter), in the Schwarzschild background, for large distances is independent of r (see Ref. 9). We expect that in equilibrium gravitational correlations behave in a similar way, even considering charged and rotating balck holes, thus we conclude that $d - 2 + \eta = 0$. Similarly, we find $\nu = 1/2$. This two new critical exponents satisfy the scaling, $(2 - \eta)\nu = \gamma$, and hyperscaling, $\nu d = 2 - \alpha$, relations, only if $d = 2$. We thus see to appear this dimensionality as the one appropriate to describe the thermodynamical properties of the black hole near criticality. It is interesting to note here that all the critical exponents found so far correspond exactly to those of the gaussian model[8] in two dimensions since it has the following critical exponents[10]

$$\alpha = 2 - d/2 \ , \ \beta = (d-2)/4 \ , \ \gamma = 1 \ , \ \delta = \frac{d+2}{d-2} \ , \ \eta = 0 \ , \ \nu = \frac{1}{2} \ . \tag{6}$$

On the other hand, if the dimensionality of the system where $d \geq 4$, the Renormalization Group analysis tell us [4,8] that the operators we could have added to the gaussian hamiltonian are "irrelevant" in the sense that they do not contribute to modify the critical exponents, which will then be those of the Gaussian model or the mean field (Landau) theory, and thus different from Eqs. (2)-(4). There is still the possibility of having $d = 3$, as is the case of most realistic system, e.g. those studied in the Laboratory. In $d = 3$, the operator ϕ^4 becomes relevant. One can make a perturbation theory based on the gaussian part of the hamiltonian and obtain a set of critical exponents [8] that fit very well with Lab experiments, but are not those we have found for black holes. We are thus left with $d = 2$ (since for $d < 2$ no critical phenomena takes place). The problem here is that *all* operators of the form ϕ^{2n} and $|\nabla\phi|^2\phi^{2n}$ are relevant and will modify the critical exponents.

We can now conclude that the first order approximation to Quantum effects in black holes correspond to the Gaussian approximation. We can also conjecture that the critical phenomena in black hole will still be present when one considers higher order corrections; that the scaling laws will continue to hold, but the critical exponents that will fulfil this laws, will probably be different from those given by Eqs. (2)-(4).

Notably, scaling during phase transitions involving gravitational effects can also be found in several scenarios such as: Inflation at late times [11,12], the strong field collapse of a massless scalar field coupled to gravity [13] and in cosmic string networks[14].

The implications of the occurrence of critical phenomena in Black holes are of great importance and makes deeper the connection among Gravitation, Field Theory and Statistical Physics. New light can be shed on the origin of the gravitational entropy and other quantum properties of black holes. For example, if we identify the two-dimensional effective surface with the event horizon and note that generalization of scaling leads to conformal invariance and then to string theory, we can arrive to 't Hooft's string interpretation of black holes[15].

The results presented in this essay lead us to consider the following effective model for dealing with a coarse grained quantization of black holes:

A black hole appears to an external observer as if it had all its quantum degrees of freedom concentrated on a thin membrane tightly covering the horizon.

This "phenomenological" model is of course observer - dependent, since in a reference system falling with the matter that will form the black hole, nothing special nor the membrane is seen when crossing the horizon. It is also clear that it is the "skin" on the horizon that we propose to consider as a system to quantize by using the standard rules of quantum field theory. In particular, we expect an unitary S - matrix to exist, and to describe the process of formation and evaporation of a black hole.

A simple way to show how the mass of a black hole should be quantized can be obtained by describing the black hole by a wave function (corresponding to the order parameter in a critical system) depending only on the two angular coordinates that cover the horizon surface. In this simplified model the only effect of the black hole gravitational field is to provide with the background geometry, i. e. the spherical surface representing the horizon. If we impose to this wave function the Klein - Gordon equation

$$\left\{-\partial_t^2 + \frac{1}{r_H^2}\nabla_\Omega^2 + \mu^2\right\}\phi(\vartheta,\varphi,t) = 0 \quad , \tag{7}$$

where $r_H = M + \sqrt{M^2 - a^2 - Q^2}$ is the horizon radius. The energy of the system is given by $E_l^2 = \mu^2 + \frac{\hbar^2 l(l+1)}{r_H^2}$; $l = 0, 1, 2, ...$ Since $r_H \simeq \frac{2GM}{c^3}$ and $E_l \simeq M$, this implies that for big l

$$M \simeq \frac{1}{\sqrt{2}} M_{pl}\sqrt{l} \quad , \quad \text{with } l \text{ a positive integer.} \tag{8}$$

This represents a quantization of the black hole mass in units of the Planck mass. It is interesting to remark that the $\sqrt{l}$ dependence have also been found by Bekenstein [16] using the quantization of adiabatic invariant action integralsand by Mukhanov[17]. The black hole radiation will now come out in the form of a line spectrum with most of the radiation at the frequency $\hbar\omega_l = \Delta M c^2 = \frac{M_{pl}^2 c^2}{4M}$ (also in multiples of this frequency), which corresponds to the maximum of the (continuum) Hawking spectrum i.e. $\hbar\omega_{max} \sim k_B T_{BH} \sim \frac{M_{pl}^2 c^2}{M}$.

Since our black hole system, as we have seen, has an associated effective dimension equal to two, the proportionality entropy - area can be expected to appear in a natural way. In fact, since the mass of the black hole is quantized there must be a finite number of internal states. They can be counted by noting [17] that a black hole of mass M given by Eq. (8) can be formed in 2^{l-1} different (and equivalent) ways from units of M_{pl}. The entropy associated with the ignorance of the exact way in which the black hole formed, can be evaluated, in a first approximation, as $S_{bh} \simeq k_B \ln [2^{l-1}]$. For large l we have

$$S_{bh} \simeq k_B \ln 2 \left(\frac{M}{M_{pl}}\right)^2 \simeq \frac{k_B l_{pl}}{4\pi} \ln 2 \left(4\pi r_H^2\right) \quad , \tag{9}$$

which gives the well - known proportionality between entropy and area of a black hole.

One should not be bewildered by these results, since they are founded on a crude approximation to the quantum black hole problem. The model is necessarily incomplete (a second quantized description should be considered, for example). Also 't Hooft suggests that the quantum states labeled by E_l in Eq. (8) are enormously degenerated[18]. It is also important to evaluate the width of each energy level (to account for the quantum instability of black holes) and compare it to the separation between energy levels. However, what we wanted to rescue from the above crude model is the relevance of the essentially two-dimensional nature of semiclassical black holes.

Thus, in conclusion, we can say that black holes may have “no hairs” [19], but instead they seem to behave as if they had a “skin”.

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