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A NEW GRAVITY METER

SUMMARY

A new system for the measurement of gravity is described in which g is compared with centripetal acceleration. An AC-null method is employed so that the measurement becomes independent of the transducer characteristic. The value of g is finally measured by the determination of a frequency. The instrument should make it possible to obtain repeatable measurements in a much shorter time than with pendulum methods. Integration methods for the elimination of the effects of periodic accelerations on a moving platform can easily be applied.

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THE NEW METHOD

For the absolute determination of gravity, methods employing Kater's reversible pendulum are usually employed. Pendulum observations are relatively slow, tedious and inaccurate, but do furnish gravity differences in milligals. (1 gal is the acceleration 1 cm per sec^2, hence 1g = 980 gal). Relative measurements are made with spring type gravity meters. These instruments furnish rapid and easy to make readings with sensitivities of .01 milligal, but the meters have to be calibrated frequently against pendulum observations.

The change of gravity with height gives us a physical picture of the required accuracy.

$$g_h \doteq g(1 - \frac{2h}{R})$$

g is the sea level value

h is the altitude in meters

R is the earth radius ($6370 \cdot 10^3$m)

for h = 3m, we obtain a reduction of g by about 1 milligal. Hence, if we lift the gravity meter from the floor to the ceiling of a room, the instrument should give a definite reading.

A new method for the determination of gravity will be described which is rapid and easy to apply and still has a sensitivity of at least 1 mgal. After the instrument has been calibrated once, measurements are repeatable without further calibration. It is hoped that future advances in the art will make a sensitivity of .1 or even .01 mgal feasible. The method should be applicable at sea and in airplanes because the output of the instrument is obtained as an AC voltage, the average of which can easily be obtained by integration.

The acceleration due to gravity is continually compared with centripetal acceleration by means of an AC null method whereby a single transducer is exposed to equal pressure amplitudes from g and from the centripetal acceleration. In this way, the transducer characteristic is rendered non-critical.

In figure 1, a long arm with length r_1 is shown, rotating around A with angular velocity ω_1 and carrying at its end a mass m_1. Around its end two short arms with length r_2 are rotating with ω_2 and $-\omega_2$ relative to the first arm, carrying masses m_2. A strain gage type transducer T is inserted in the long arm, generating a voltage proportional to the radial stress existing in the arm. The axles through A and B are horizontal. The rotations are produced by motors and maintained at constant angular velocities.

Let $|\omega_1|$ be equal to $|\omega_2|$. This means that, while the long arm performs one revolution, the short arms revolve once around the long arm. As shown in figure 2, the phase of the revolutions is adjusted in such a way that, for the long arm pointing vertically up, the two masses m_2 are together at their highest point. While the long arm points vertically down, the two masses are again together and pointing upwards.

One can guess intuitively how the system works: For the long arm pointing up, gravity compresses the transducer. For the arm pointing down, gravity expands the transducer. The centrifugal force, however, works just in the opposite direction. The long arm is lengthened by the short arms in its up position and shortened in its down position so that the centrifugal force expands the transducer more in the up position than in the down position. For a certain value of m_1 and m_2 and for a certain $\omega = \omega_1 = \omega_2$, complete cancellation of the AC output occurs so that g can be determined by measuring ω.

The transducer together with a high gain amplifier and detector are used as a null indicator. Only two elements have to be controlled with high accuracy, namely ω and r_2. This will be explained later after the equations of motion have been established and after an error analysis has been performed.

ANALYSIS

The design as shown in figure 1 is just one special arrangement. In order to keep the treatment as general as possible, it will be assumed that $\omega_1 \neq \omega_2$. The problem is defined by the determination of the acceleration of a particle P on a moving curve. This acceleration is the resultant of:

a_1 - The acceleration of constraint

a_2 - The relative acceleration (found as if the curve were at rest)

a_3 - The Coriolis acceleration (Twice the product of the relative velocity of the particle along the curve by the angular velocity of the curve)

a_1 - The acceleration of constraint of any point P on a moving curve is the resultant of:

a_{11} - The acceleration of point A around which the curve rotates

a_{12} - $r\omega^2$ from P to A, due to the rotation of AP about A.

a_{13} - $r\dot{\omega}^2$ perpendicular to AP.

Assuming that only one mass m_2 is rotating and further assuming that ω_1 and ω_2 are uniform and that all elements are rigid (no bending of the arms), we can determine the radial acceleration along r_1: (the centripetal accelerations are assumed positive for directions towards A and, for $t = 0$, the initial position of m_2 is at its highest point).

For mass m_2:

$$a_{11} = 0$$

$$a_{12} = \omega_1^2(r_1 + r_2 \cos \omega_2 t)$$

$$a_{13} = 0$$

$$a_1 = \omega_1^2(r_1 + r_2 \cos \omega_2 t)$$

$$a_2 = \omega_2^2 \, r_2 \cos \omega_2 t$$

$$a_3 = 2\omega_1 \omega_2 \cos \omega_2 t$$

$$a_r = \omega_1^2 \, r_1 + (\omega_1 + \omega_2)^2 \, r_2 \cos \omega_2 t$$

For mass m_1:

$$a_r = a_1 = \omega_1^2 r_1 \qquad (a_2 = 0 \ , \ a_3 = 0)$$

For two masses m_2 counter rotating, the Coriolis term a_3 disappears because of mutual cancellation of the two accelerations. We obtain for m_2 and m_2':

$$a_r = \omega_1^2 \, r_1 + (\omega_1^2 + \omega_2^2) \, r_2 \cos \omega_2 t.$$

It is interesting to show that, if radial acceleration is considered, the function of two counter-rotating masses m_2 is equivalent to the function of a single mass $2m_2$, swinging radially along the large arm with sinusoidal motion as shown in figure 3.

$$s = r_2 \cos \omega_2 t \qquad ; \qquad \dot{s} = - \omega_2 \, r_2 \sin \omega_2 t$$

$$\ddot{s} = - \omega_2^2 \, r_2 \cos \omega_2 t.$$

Because we assume the acceleration towards A as positive, the relative acceleration of 2m is:

$$a_2 = \omega_2^2 \; r_2 \cos \omega_2 t.$$

Hence:

$$a_{11} = 0$$

$$a_{12} = \omega_1^2 \; (r_1 + r_2 \cos \omega_2 t)$$

$$a_{13} = 0$$

$$a_2 = \omega_2^2 \; r_2 \cdot \cos \omega_2 t$$

$$a_3 = 0$$

$$a_r = \omega_1^2 \; r_1 + (\omega_1^2 + \omega_2^2) \; r_2 \cos \omega_2 t$$

This is the same value we obtained before for two counter-rotating masses.

We are now ready to write the equation for the radial force in our idealized (not bending) system.

The sign for the forces directed outward from A are assumed to be positive.

$$F_r = 2m_2[\omega_1^2 r_1 + (\omega_1^2 + \omega_2^2) \; r_2 \cos \omega_2 t] + m_1 \; \omega_1^2 \; r_1$$

$$-(m_1 + 2m_2) \; g \cos \omega_1 t = F_{DC} + F_C + F_g.$$

In figure 4, F_r is plotted versus time. It consists of three parts: F_{DC}, a constant force, F_C at frequency $\frac{\omega 2}{2\pi}$ and F_g at frequency $\frac{\omega 1}{2\pi}$. F_{DC} and F_C only would exist if the axes were vertical. F_g is caused by gravity. For clarity, it is assumed that $\omega_1 < \omega_2$. The value of gravity could now be determined if F_C is compared with F_g. In order to compare two voltages with different frequencies, however, rectifiers are necessary. In addition, the same transducer sensitivity would be required for ω_1 and ω_2.

These drawbacks can be avoided by making $\omega_1 = \omega_2$ so that the transducer is driven by two periodic forces with the same frequency and the same amplitudes with opposite phases. A real null method can now be applied for the measurement of gravity.

If $\omega = \omega_1 = \omega_2$ and if only AC voltages with the frequency $\frac{\omega}{2\pi}$ are derived from the transducer, we obtain:

$$E = K \left[4m_2 \omega^2 r_2 - (m_1 + 2m_2) g\right] \cos \omega t \qquad (1)$$

K is the voltage produced by the transducer per unit force.

For E = 0, we obtain:

$$g = \frac{4m_2 \omega^2 r_2}{(m_1 + 2m_2)} \text{ cm/sec}^2.$$

r is measured in cm and ω in radians per sec.

We can now determine the required accuracy of the components for a non-bending system. If g has to be measured with an accuracy of 1 part in 10^6, m_1, m_2 and r_2 have to be stable to 1 part in 10^6. ω has to be controlled to one part in $2 \cdot 10^6$ because $\frac{\Delta g}{g} = 2\frac{\Delta \omega}{\omega}$. It should be noted that the g-reading is independent of r_1. For a non-rigid system, it will be shown that r_1 has to be controlled to 4 parts in 10,000.

For a g of 980 cm/sec^2, the equation supplies also the correct relation between ω_1, r_2, m_1 and m_2:

$$\omega^2 r_2 = 245 \frac{m_1 + 2m_2}{m_2}$$

One set of constants could be chosen as follows:

$$\omega = 2\pi \cdot 100 = 628 \qquad (\omega^2 = 394400)$$

$r_2 = 1$ cm

$m_1 = 1608$ grams

$m_2 = 1$ gram

This choice has two drawbacks:

1) r_2 is small so that the dimension of the bearing is a large part of the arm. Hence, an accuracy of r_2 of one part in 10^6 is not easily obtainable.

2) The radial force exerted by the masses $2m_2$ has a maximum value of:

$$F_m' = 2m_2\,\omega^2\,(r_1 + 2r_2) \text{ dynes}$$

For $r_1 = 10$ cm, $F_m = 9.5 \cdot 10^6$ dynes

$$F_m = 9.5 \cdot 10^6 \frac{1}{g} = 9720 \text{ gram force.}$$

This load acts on the bearings of the small arm and makes it impossible to maintain r_2 with the required accuracy.

A reduction of the angular velocity to $\omega = 2\pi \cdot 10$ and an increase of r_2 to $r_2 = 10$ cm seems to be a better choice. r_1 is increased to 30 cm.

$\omega = 2\pi \cdot 10 = 62.8$ $\qquad (\omega^2 = 3944)$

$r_2 = 10$ cm

$m_1 = 159$ gram

$m_2 = 1$ gram

For $r_1 = 30$ cm, $F_m = 394000$ dynes

$F_m = 403$ gram force.

The ratio of r_2 to the bearing diameter has been increased by a factor of 10. In addition, the load on the bearing is reduced. It is believed, that under these conditions, the required accuracy can be maintained.

THE INSTRUMENT

Figure 7 shows a schematic design of the apparatus. The small masses m_2 and m_2' are carried by two thin aluminum disks. For the control of speed these disks support on their periphery a magnetic recording signature of 100000 waves. Assuming a radius of 11 cm, the circumference of the disks is 69 cm. The recorded wavelength is, therefore, $.69 \cdot 10^{-3}$ cm, a value which is actually used in good recorders. The disks are driven by two motors M and M' with 10 revolutions per second. Two magnetic heads H and H' are arranged on the long arm.

The speed control of the long arm is carried out the same way. A disk D with magnetic recording on its rim is coupled to the arm. Magnetic head K is employed to pick up the recorded wave. Hence, while both arms rotate with 10 rps, the heads will deliver waves with a frequency of 1 Mc/sec. Disk D carries an additional magnetic signature on its surface which produces in the magnetic head L a wave with a frequency of 10 cps. The long arm together with disk D are driven by motor N.

Figure 8 shows a block diagram of the servo-loops. Motors M and M' are slaved to motor N by comparing the waves from K with the waves from H and H' in the phase detectors P and P'. Whenever M or M' advance or retard with respect to N, a positive or negative voltage is developed in P and P'. These voltages are employed in the motor control devices MC and MC' to produce exact speed and phase synchronism between N and M and M'. The slaving operation could, of course, be performed mechanically. It is, however, believed that an electronic system is more exact and has the advantage that less weight has to be supported by the moving parts.

The 10-cycle wave derived from transducer T is used to control the speed of motor N. Inspecting equation 1, it can be seen that the expression for E consists of two parts: The centrifugal wave and the gravity wave. For $E = 0$, their amplitudes are identical and, if ω is measured, g can be determined. If ω is too high or too low, the phase of E will be either 0° or 180° compared with the phase of the gravity wave alone. This phase reference is delivered by transducer L. The amplified wave from T is compared with the output from L in the phase-sensitive rectifier Q. If its output is positive or negative, N is speeded up or slowed down by action of motor control NC.

Finally, the 1 Mc wave from the magnetic head K is compared with the output from a master clock CL in the mixer MI. The difference frequency is measured in the counter CC. Because $\frac{\Delta g}{g} = 2\frac{\Delta\omega}{\omega}$ and assuming that the master clock delivers 1 Mc and that this frequency corresponds to a certain g, a deviation of g by 1 milligal will produce a beatnote of $\frac{1}{2}$ cps. Observation of this beatnote for a period of 10 seconds should be sufficient to determine the exact value of the beat-frequency.

Actually, it will take longer to perform a measurement of g. The output of T will contain a certain amount of noise due to an unavoidable amount of bearing-roughness so that the bandwidth of amplifier A has to be restricted. The same effect can be achieved by introducing a low pass filter after the phase-sensitive detector Q. Whichever method is used, the time of response of the instrument will be increased. If the instrument is used on a periodically accelerated platform, additional integration networks have to be applied after Q.

Variations of the torque that the motor N has to deliver are reduced by extending the long arm to the other side and by the addition of 2 more motors for the rotation of two additional small masses as explained in the appendix. These motors are driven in phase synchronism with M and M'

Figure 9 shows the transducer mounting.* It is very important that the transducer should respond only to radial forces. It should be as insensitive as possible to any lateral stress of the arm. The long arm consists of parts 1 and 2. They are connected by means of thin flexible membranes 3, permitting free radial but impeding lateral motions. The transducer is connected between part 1 and 2' which is connected to part 2 by a flexible part 4. Part 2' can again be subdivided into parts 2", 2"', etc. This way, a "filter" is formed, eliminating any undesired bending of the transducer by a tilting action of part 2.

The alignment of the instrument, i.e., any deviation from the horizontal is not critical. Let us assume that the instrument is tilted about the axis of the large arm by an angle α. The centrifugal wave will now be phase shifted by α with respect to the gravity wave. A phasor diagram is shown in

*Suggested by Mr. B. M. Horton.

figure 10. It is assumed that for correct alignment $P = K\ 4m_2\ \omega^2\ r_2 = K(m_1 + 2m_2)g$. (See equation 1). After the two waves have been compared in the phase sensitive rectifier Q, an error voltage ΔE will be produced which is proportional to $P(1 - \cos\alpha) \doteq P \cdot \frac{\alpha^2}{2}$.

For an accuracy of 1 part in 10^6, ΔE has to be kept smaller than $P \cdot 10^{-6}$.

Hence:

$$P \frac{\alpha^2}{2} < P \cdot 10^{-6} \quad \text{and} \quad \alpha < \sqrt{2} \cdot 10^{-3} \text{ radians.}$$

This means that α has to be smaller than 25 arc seconds, an adjustment which can easily be performed by conventional means.

APPENDIX

THE BENDING OF THE LARGE ARM

Because the new method is based on the comparison of two AC-voltages, their waveforms should be as free from distortion as possible. This is indeed the case if all elements of the apparatus are rigid. In a practical machine, however, bending of the arms will occur, causing phase-modulation of the angular velocity. Again, in order to keep the equations as general as possible, it will be assumed that ω_1 and ω_2 are not equal.

The moment of inertia I of the total system changes periodically with frequency $\frac{\omega 2}{2\pi}$ because of the rotation of the small arms. The angular momentum tends to remain constant so that ω_1 will be modulated with frequency $\frac{\omega 2}{2\pi}$. If the motor would drive the long arm through a heavy flywheel and if the shaft between flywheel and arm were not flexible, the arm would be driven with essentially uniform velocity.

Now, the flywheel and the unflexible shaft can be replaced by an arrangement as shown in figure 5. By extending the long arm to the other side of the main bearing and by rotating two additional masses as shown, the angular momentum will remain constant.

For 2 small masses as shown in figure 1:

$$(\omega_1 I)_{2m_2} = \omega_1 \cdot 2m_2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos \omega_2 t)$$

For 4 small masses as shown in figure 5:

$$(\omega_1 I)_{4m_2} = \omega_1 \cdot 2m_2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos \omega_2 t + r_1^2 + r_2^2 - 2r_1 r_2 \cos \omega_2 t)$$

$$(\omega_1 I)_{4m_2} = \omega_1 \cdot 4m_2 (r_1^2 + r_2^2).$$

It can be seen that the angular momentum is constant.

Even if the long arm is driven with uniform velocity, the end of the long arm would bend back and forth at $\frac{\omega_2}{2\pi}$. It will be shown, however, that the effect on the accuracy is so small that it can be neglected.

An iterative process will be used for the computation:

Step 1: The long arm is driven with ω_1.

Due to the ω_2 rotation of the small arms, tangential forces acting on point B are produced, bending the long arm periodically with an angle excursion γ and causing a new angular velocity $\omega(t)$ of point B. This $\omega(t)$ is a sinusoidally phase-modulated wave.

Step 2: Point B rotates with ω and not ω_1. The tangential forces acting on B are recalculated and lead to a new time varying bending angle. The phase excursions are modified and a new ω' of point B is determined. This $\omega'(t)$ is now a non-sinusoidally phase-modulated wave.

Step 3: Point B rotates with ω' and not ω. The process could be continued indefinitely, but it will be shown that, because of the smallness of the perturbations, the process can be terminated after step 2.

The only tangential acceleration to be considered in step 1 is the Coriolis term (a_3). In step 2, the acceleration term (a_{13}) has to be added.

1) $$a_{t_1} = -2\omega_1 \omega_2 r_2 \sin \omega_2 t$$

The acceleration is assumed to be positive in clockwise direction.

The force acting on the long arm is:

$$F_{t_1} = +2m_2 \cdot 2\omega_1 \omega_2 r_2 \sin \omega_2 t$$

F_{t_1} can also be computed by deriving the torque which is the time derivative of the angular momentum $\omega_1 J$. The torque acting on the system is:

$$T = \frac{d}{dt}\omega_1 I = \frac{d}{dt}\omega_1 \cdot 2m_2(r_1^2 + r_2^2 + 2r_1 r_2 \cos\omega_2 t).$$

$$T = -4m_2 \omega_1 \omega_2 r_1 r_2 \sin\omega_2 t.$$

Hence, the tangential force acting on B is:

$$F_{t_1} = \frac{T}{r_1} = 4m_2 \omega_1 \omega_2 r_2 \sin\omega_2 t.$$

For: $m_2 = 1$ gram, $\omega = \omega_1 = \omega_2 = 2\pi \cdot 10$ and

$r_2 = 10$ cm

$F_{t_1}^{max} = 158000$ dynes $\doteq 160$ gram force.

Assuming that this force acts on a beam with a length $l = 20$ cm (2/3 or r_1), the bending angle γ can be computed. A beam with rectangular cross-section (a = b = 3) is assumed although a tubular design with similar stiffness will probably be chosen for a practical design. The deflection

$$h = \frac{4}{E}\frac{l^3}{a^3 b} F_{t_1} = 3 \cdot 10^{-5} \text{ cm}.$$

E is Young's modulus ($2 \cdot 10^6$ Kg/cm^2 for steel)

$$\gamma = \frac{h}{l} = 1.5 \cdot 10^{-6} \text{ radians.}$$

Actually, the bending angle will be smaller because the constant centrifugal force increases the effective stiffness of the beam. A value of $\gamma = 10^{-5}$, however, will be assumed for further computations in order to accommodate any additional bending which may be caused by the transducer mounting. Hence, the long arm is periodically bent back and forth with a maximum angle $\gamma = 10^{-5}$ radians.

$$\phi_1 = \omega_1 t + \gamma \sin \omega_2 t$$

$$\omega = \frac{d\phi}{dt} = \omega_1 + \omega_2 \gamma \cos \omega_2 t$$

$$\dot{\omega} = -\omega_2^2 \gamma \sin \omega_2 t$$

2) In the second step, the accelerations not only of m_2 but also of m_1 have to be considered.

For m_2 :

$$a_{t_2}^{m_2} = -2\omega\omega_2 r_2 \sin\omega_2 t + \dot{\omega}\,\frac{r_1^2 + r_2^2 + 2r_1 r_2 \cos\omega_2 t}{r_1}$$

$$\text{for } r_1 > r_2 : \quad \frac{r_1^2 + r_2^2 + 2r_1 r_2 \cos\omega_2 t}{r_1} \doteq r_1 + 2r_2\cos\omega_2 t.$$

$$a_{t_2}^{m_2} = -2\omega\omega_2 r_2 \sin\omega_2 t + \dot{\omega}(r_1 + 2r_2\cos\omega_2 t).$$

$$a_{t_2}^{m_2} = -2\omega_1\omega_2 r_2 \sin\omega_2 t - 2\omega_2^2\gamma r_2 \sin\omega_2 t\cos\omega_2 t$$
$$- \omega_2^2\gamma r_1 \sin\omega_2 t - 2\omega_2^2\gamma r_2 \sin\omega_2 t \cos\omega_2 t.$$

$$a_{t_2}^{m_2} = -2\omega_1\omega_2 r_2 \sin\omega_2 t - \omega_2^2\gamma r_1 \sin\omega_2 t$$
$$- 2\omega_2^2\gamma r_2 \sin 2\omega_2 t.$$

$$F_{t_2}^{m_2} = 2m_2 a_{t_2}^{m_2}.$$

For m_1: $$a_{t_2}^{m_1} = -\omega_2^2\gamma r_1 \sin\omega_2 t.$$

$$F_{t_2}^{m_1} = m_1 a_{t_2}^{m_1}.$$

The total tangential force is:

$$F_{t_2} = F_{t_2}^{m_2} + F_{t_2}^{m_1} = 4m_2\omega_1\omega_2 r_2 \sin\omega_2 t + (2m_2 + m_1)\omega_2^2\gamma r_1 \sin\omega_2 t$$
$$+ 4m_2\omega_2^2\gamma r_2 \sin 2\omega_2 t.$$

Compared with F_{t_1} (the first term in above expression), the two last terms are reduced by $\gamma = 10^{-5}$ and can be neglected. The iterative process can be terminated after the second step.

A continuation would follow these lines:

$$\phi_2 = \omega_1 t + K\, F_{t2}(t)$$

$$\omega' = \frac{d\,\phi_2}{dt} = \omega_1 + K \frac{d\,F_{t2}}{dt} \quad \text{etc.}$$

THE TORQUE EXERTED ON THE SMALL ARMS

The torque exerted on each short arm is caused by accelerations of m_2, perpendicular to r_2. The only term to be considered is a_{12}, the acceleration of m_2 due to its rotation around A. There is no perpendicular component of the terms a_2 and a_3.

Figure 6 shows the vector diagram.

$$a_{12} = \omega_1^2 \ell .$$

$$\frac{\ell}{\sin \omega_2 t} = \frac{r_1}{\sin(\omega_2 - \omega_1)t} .$$

$$\ell = r_1 \frac{\sin \omega_2 t}{\sin(\omega_2 - \omega_1)t} .$$

$$a_{12} = \omega_1^2 r_1 \frac{\sin \omega_2 t}{\sin(\omega_2 - \omega_1)t} .$$

$$a_p = a_{12} \sin(\omega_2 - \omega_1)t = \omega_1^2 r_1 \sin \omega_2 t .$$

$$F_p = -a_p m_2 = -m_2 \omega_1^2 r_1 \sin \omega_2 t .$$

For $m_2 = 1$ gram, $\omega_1 = 2\pi \cdot 10$ and $r_1 = 10$ cm.

$F_p^{max} = 39500$ dynes $= 40.5$ gram force.

The torque $T^{max} = F_p \cdot r_2 = 405$ gram cm.

This value is so low that the bending angle of the small arm can be neglected. Still, the torque has to be considered for the design of the motor, driving the two small arms so that excessive periodic phase excursions can be avoided. Second order effects caused by $\dot{\omega}$, are so small that they do not have to be considered.

THE EFFECT OF BENDING OF THE LARGE ARM ON THE RADIAL CENTRIFUGAL FORCES

It was shown before that, because of the Coriolis acceleration, the angular velocity of B is not uniform. Hence, in order to compute the radial forces with greater accuracy, ω_1 has to be replaced by $\omega(t)$.

$$\omega = \omega_1 + \omega_2 \gamma \cdot \cos \omega_2 t .$$

The analysis of the stiff system has led to the equations:

For masses m_2:

$$a_{r_{ot}}^{m_2} = \omega_1^2 r_1 + (\omega_1^2 + \omega_2^2) r_2 \cos \omega_2 t$$

For mass m_1:

$$a_{r_{ot}}^{m_1} = \omega_1^2 r_1 .$$

Replacing ω_1 by ω we obtain:

For masses m_2:

$$a_r^{m_2} = r_1(\omega_1^2 + \omega_2^2 \gamma^2 \cos^2 \omega_2 t + 2\omega_1 \omega_2 \gamma \cos \omega_2 t) + r_2(\omega_1^2 + \omega_2^2 \gamma^2 \cos^2 \omega_2 t + 2\omega_1 \omega_2 \gamma \cos \omega_2 t) \cos \omega_2 t + r_2 \omega_2^2 \cos \omega_2 t .$$

because: $\sin^2 \alpha = \frac{1}{2} - \frac{\cos 2\alpha}{2}$; $\cos^2 \alpha = \frac{1}{2} + \frac{\cos 2\alpha}{2}$;

$$\cos^3 \alpha = \frac{\cos 3\alpha}{4} + \frac{3 \cos \alpha}{4} .$$

$$a_r^{m_2} = r_1 (\omega_1^2 + \tfrac{1}{2}\omega_2^2 \gamma^2 + \tfrac{1}{2}\omega_2^2 \gamma^2 \cos 2\omega_2 t + 2\omega_1 \omega_2 \gamma \omega_2 t +$$

$$+r_2(\omega_1^2\cos\omega_2 t + \tfrac{1}{4}\omega_2^2\gamma^2\cos 3\omega_2 t + \tfrac{3}{4}\omega_2^2\gamma^2\cos\omega_2 t +$$
$$+\omega_1\omega_2\gamma + \omega_1\omega_2\gamma\cos 2\omega_2 t) +$$
$$+ r_2\omega_2^2\cos\omega_2 t .$$

The DC term: $$r_1\omega_1^2 + \tfrac{1}{2} r_1\omega_2^2\gamma^2 + r_2\omega_1\omega_2\gamma .$$

The term at the fundamental frequency:

$$\cos\omega_2 t\left[r_2(\omega_1^2+\omega_2^2) + \tfrac{3}{4} r_2\omega_2^2\gamma^2 + 2 r_1\omega_1\omega_2\gamma\right].$$

At the second harmonic:

$$\cos 2\omega_2 t\left(\tfrac{1}{2} r_1\omega_2^2\gamma^2 + r_2\omega_1\omega_2\gamma\right).$$

At the third harmonic:

$$\cos 3\omega_2 t\left(\tfrac{1}{4} r_2\omega_2^2\gamma^2\right).$$

For mass m_1:

$$a_r^{m_1} = r_1(\omega_1^2 + \omega_2^2\gamma^2\cos^2\omega_2 t + 2\omega_1\omega_2\gamma\cos\omega_2 t)$$
$$a_r^{m_1} = r_1(\omega_1^2 + \tfrac{1}{2}\omega_2^2\gamma^2 + \tfrac{1}{2}\omega_2^2\gamma^2\cos 2\omega_2 t +$$
$$+ 2\omega_1\omega_2\gamma\cos\omega_2 t).$$

The DC term: $$r_1\omega_1^2 + \tfrac{1}{2} r_1\omega_2^2\gamma^2 .$$

At fundamental frequency: $$\cos\omega_2 t\,(2 r_1\omega_1\omega_2\gamma).$$

At the second harmonic: $$\cos 2\omega_2 t\left(\tfrac{1}{2} r_1\omega_2^2\gamma^2\right).$$

Neglecting all terms containing γ^2, we finally obtain the radial forces: (radial forces are assumed to be positive in outward direction).

$$F_{DC} = (m_1 + 2m_2) r_1 \omega_1^2 + 2 m_2 r_2 \omega_1 \omega_2 \gamma .$$

$$F_{\omega_2} = \cos \omega_2 t \left[2 m_2 r_2 (\omega_1^2 + \omega_2^2) + (2m_1 + 4m_2) r_1 \omega_1 \omega_2 \gamma \right]$$

$$F_{2\omega_2} = \cos 2\omega_2 t \, (2 m_2 r_2 \omega_1 \omega_2 \gamma) .$$

F_{DC} and $F_{2\omega_2}$ are rejected by frequency selective means.

$$\text{for } \omega = \omega_1 \approx \omega_2 : \; F_{\omega_2}^{max.} = 4\omega^2 m_2 r_2 + 2\omega^2 (2m_2 + m_1) r_1 \gamma .$$

For m_1 = 159 gram, m_2 = 1 gram, r_1 = 30 cm, r_2 = 10 cm and $\gamma = 10^{-5}$, the second part of this excursion is 400 times smaller than the first part. Hence, for a required accuracy of 1 part in 10^6, the second part has to be kept stable with an accuracy of 400 parts in 10^6 or 4 parts in 10^4. This means that if the bending of the large arm is considered, r_1 and γ have to be kept within four parts in 10000.

THE EFFECT OF BENDING OF THE LARGE ARM ON THE RADIAL GRAVITY FORCE

It has been shown that, for the rigid system, the gravity force is:

$$F_G = (m_1 + 2m_2) \; g \cos \omega_1 t.$$

$$a_g = g \cos \omega t(t) .$$

It was also shown that, for the system with the bending arm:

$$\omega = \omega_1 + \omega_2 \gamma \cos \omega_2 t .$$

$$\Phi = \int_0^t \omega \, dt = \omega_1 t + \gamma \sin \omega_2 t .$$

$$a_g = g \cos(\omega_1 t + \gamma \sin \omega_2 t) .$$

because: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$a_g = g\left[\cos\omega_1 t\cos(\gamma\sin\omega_2 t) - \sin\omega_1 t\sin(\gamma\sin\omega_2 t)\right].$$

$$\cos(\gamma\sin\omega_2 t) = J_0(\gamma) + 2J_2(\gamma)\cos 2\omega_2 t + 2J_4(\gamma)\cos 4\omega_2 t + \ldots$$

$$\sin(\gamma\sin\omega_2 t) = 2J_1(\gamma)\sin\omega_2 t + 2J_3(\gamma)\cos 3\omega_2 t + \ldots$$

(J_n (γ) is the Bessel function of the first kind of the nth order. γ is the argument of the Bessel function in question.)

$$a_g = g\left[J_0(\gamma)\cos\omega_1 t + 2J_2(\gamma)\cos\omega_1 t\cos 2\omega_2 t - \right.$$
$$\left. - 2J_1(\gamma)\sin\omega_1 t\sin\omega_2 t - 2J_3(\gamma)\sin\omega_1 t\cos 3\omega_2 t\right]$$

because: $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$a_g = g\left[J_0(\gamma)\cos\omega_1 t - J_1(\gamma)\cos(\omega_1-\omega_2)t + J_1(\gamma)\cos(\omega_1+\omega_2)t\right.$$
$$+ J_2(\gamma)\cos(\omega_1-2\omega_2)t + J_2(\gamma)\cos(\omega_1+2\omega_2)t$$
$$- J_3(\gamma)\cos(\omega_1-3\omega_2)t + J_3(\gamma)\cos(\omega_1+3\omega_2)t.$$

For a small argument, the following approximations can be made:

$$J_0(\gamma) = 1 - \frac{\gamma^2}{4};$$
$$J_1(\gamma) = \frac{\gamma}{2}; \quad J_2(\gamma) = \frac{\gamma^2}{8};$$
$$J_3(\gamma) = \frac{\gamma^3}{64}.$$

Only the fundamental AC component of the transducer voltage is utilized. The bending of the arm produces a reduction of this voltage by the factor $J_0(\gamma)$.

For a γ of 10^{-5}, this reduction is $\frac{\gamma^2}{4} = \frac{10^{-10}}{4}$ and represents an error in the reading of gravity of 2.5 parts in 10^{11}.

Hence, the bending of the long arm will practically not modify the transducer output. As shown before, this output consists of a wave with a frequency of 10 cps. $(\omega = \omega_1 = \omega_2)$.

ACKNOWLEDGMENT

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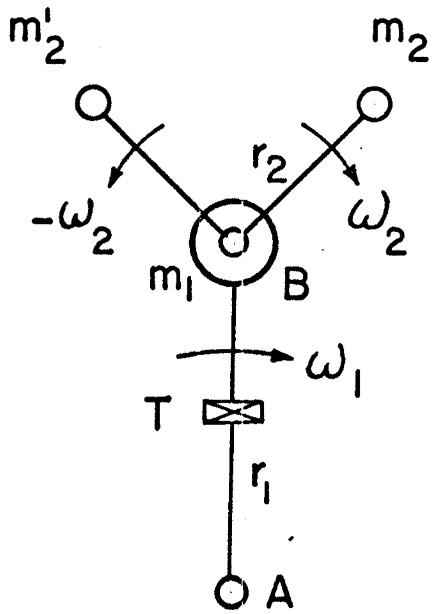


FIGURE 1

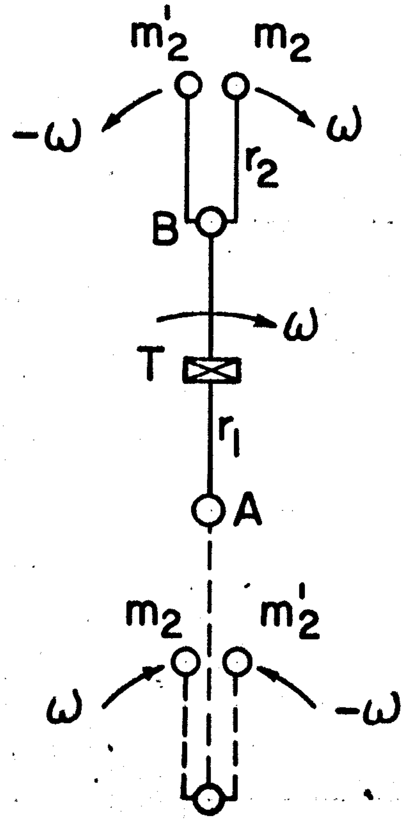
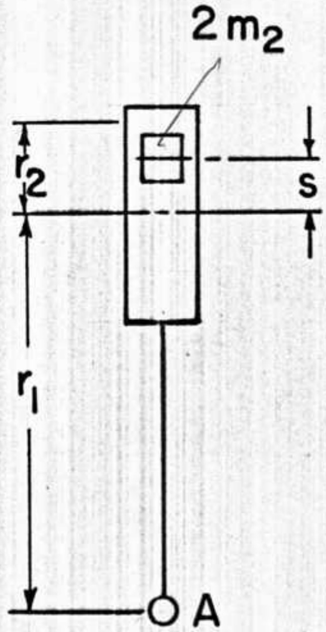


FIGURE 2



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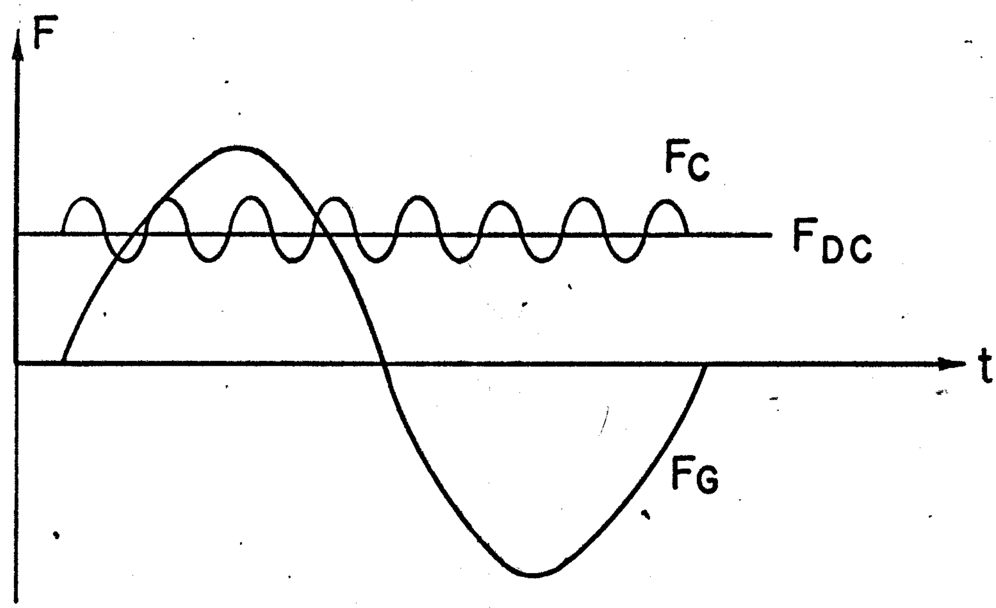

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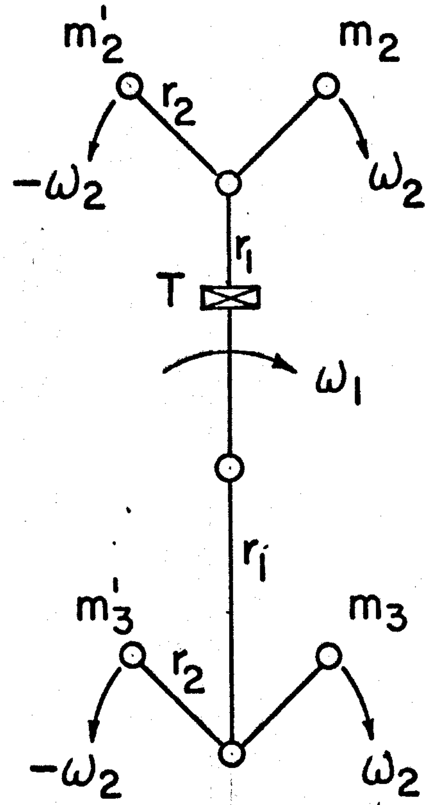

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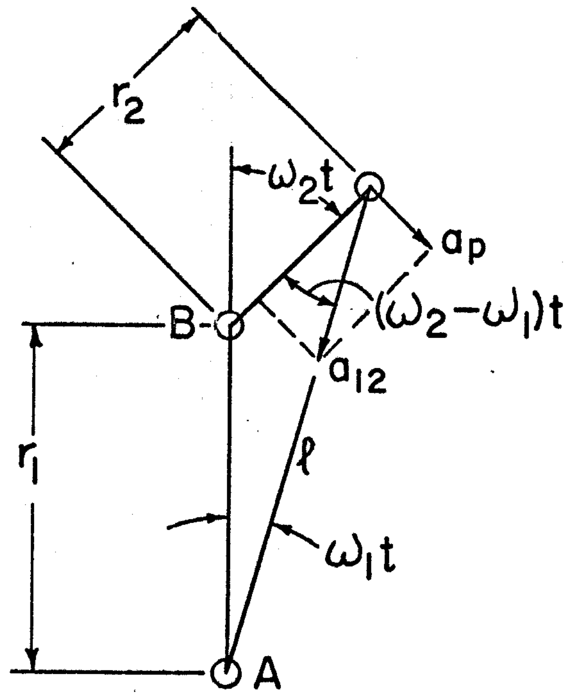

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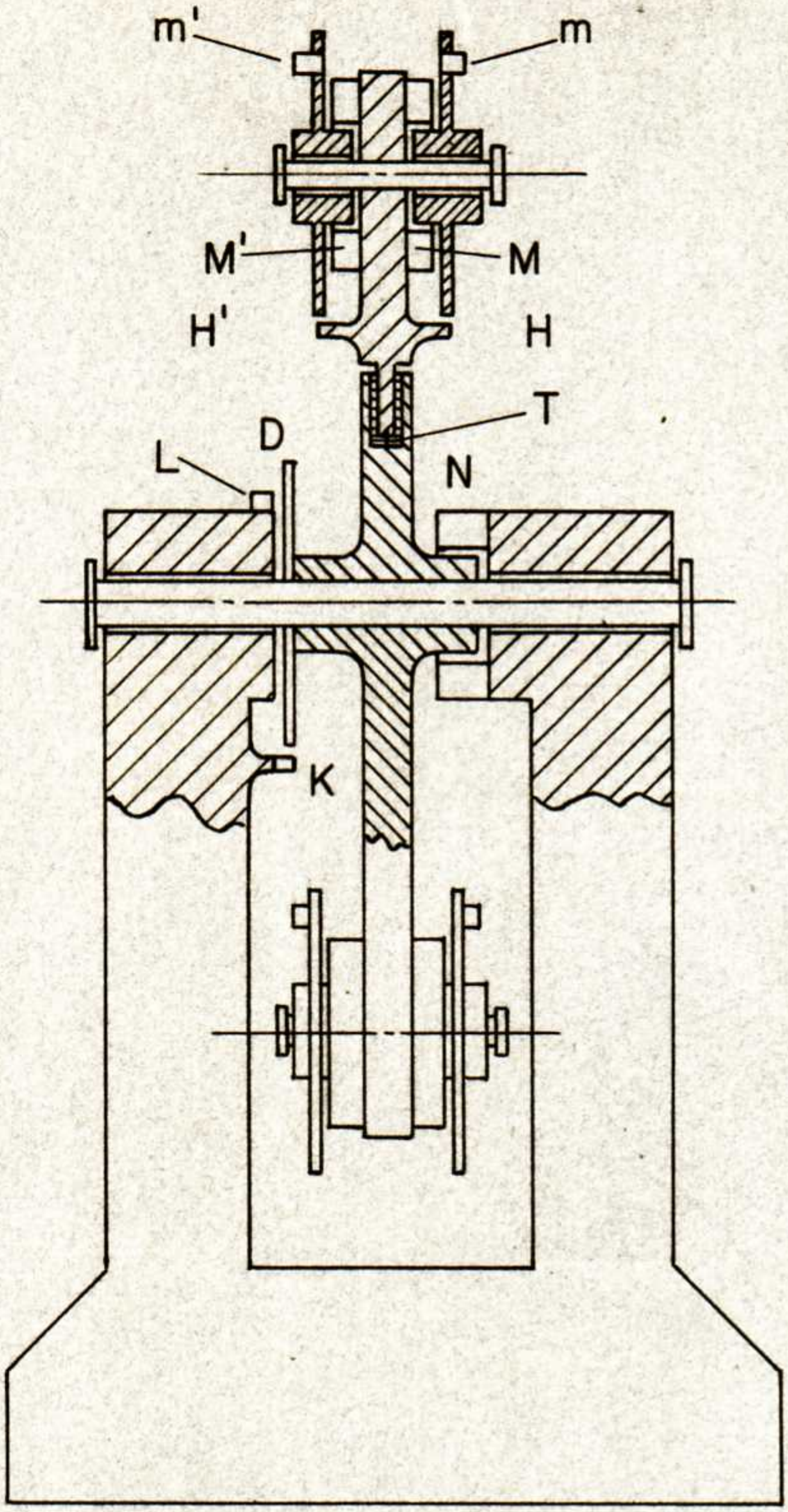
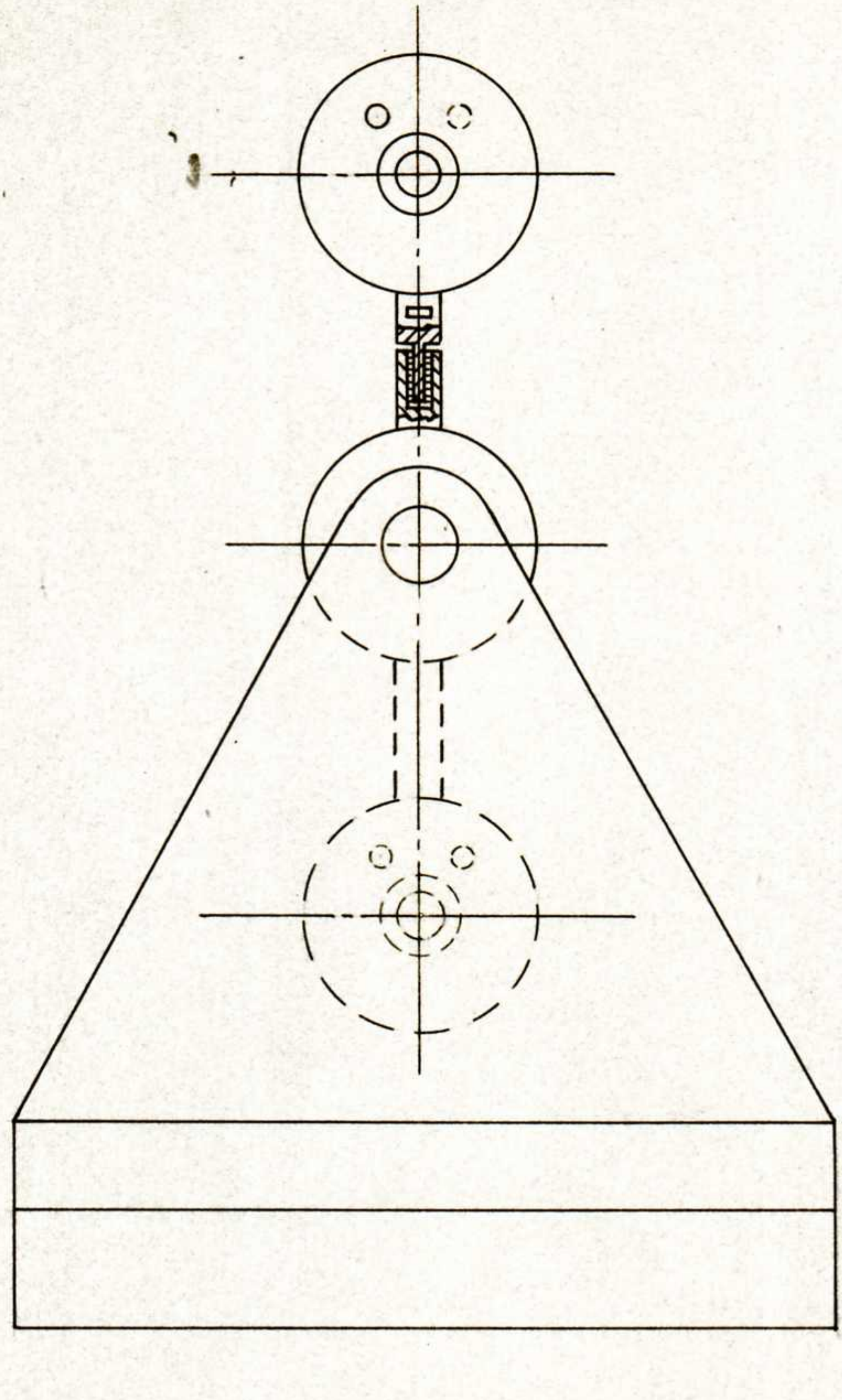
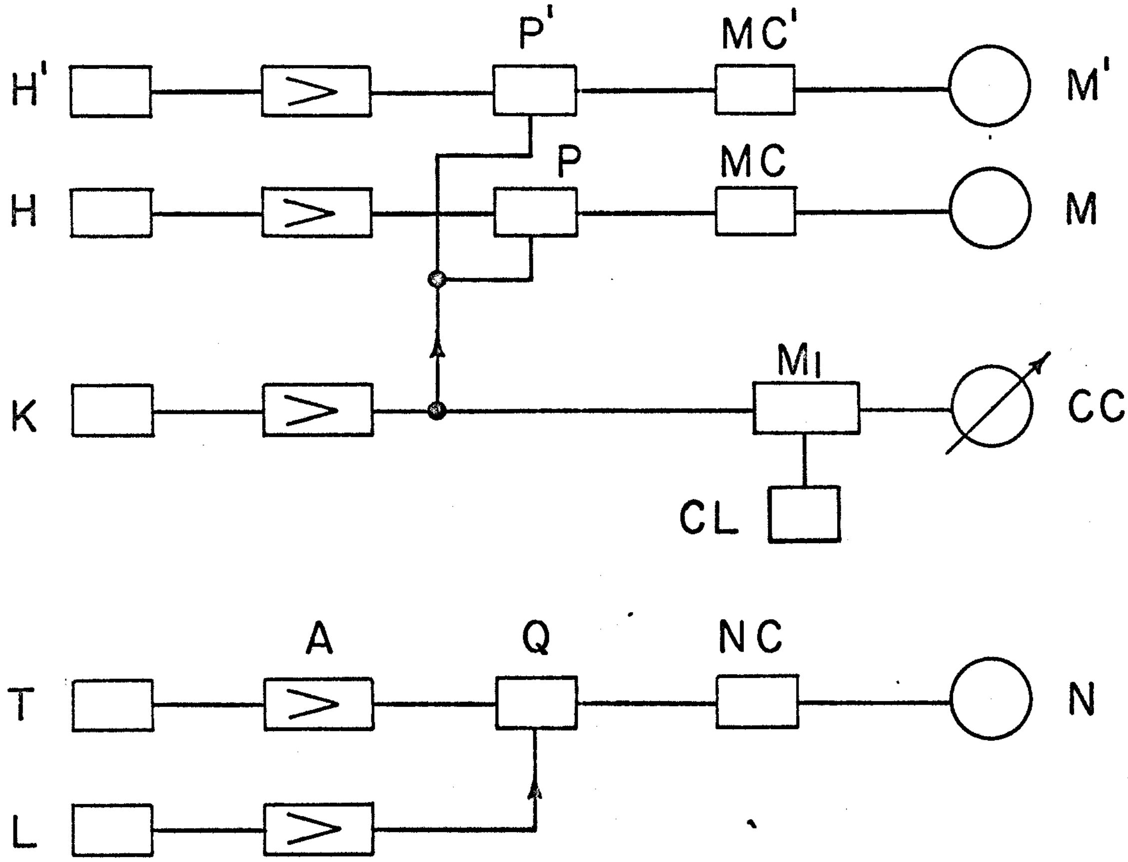


FIGURE





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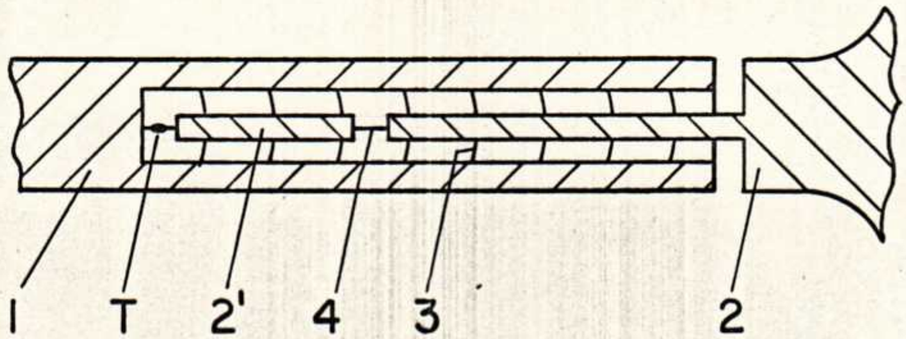

FIGURE 9

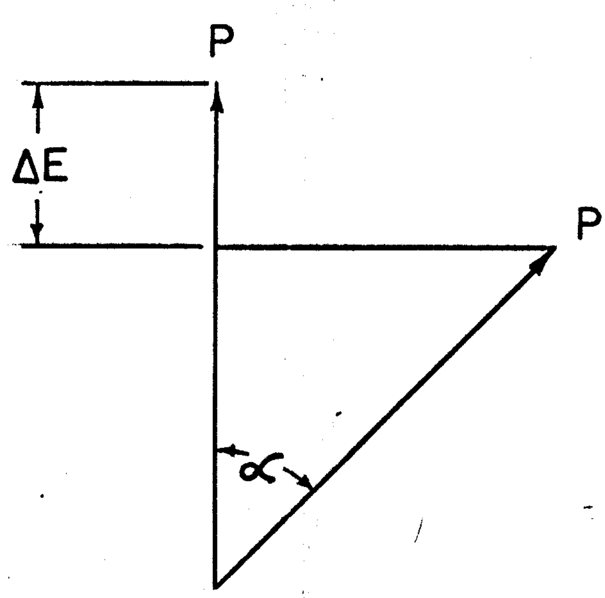

FIGURE 10