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BLACK HOLES AREN'T BLACK

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Abstract.

It is shown that quantum effects cause a black hole to radiate like a body with temperature of the order of $10^{26} M^{-1}$ ^{o}K where M is the mass of the black hole in grams. This thermal radiation means that black holes have a finite life of the order of $10^{-28} M^{3}$ secs.

The classical theory of general relativity predicts that when a body of mass M collapses to within the Schwarzschild radius $2GM/c^2$, the gravitational field becomes so strong that no further light can escape. (fig 1) An observer at infinity would never actually see the body fall within the Schwarzschild radius. Instead, it would appear to him to slow down and hover just outside while the redshift increased and the luminosity decreased exponentially. In a few milliseconds all that remains is an object which is effectively completely black but which still possesses the same gravitational field as the body that collapsed. Such an object is called a "black hole" and its boundary, the boundary of the region of space-time from which nothing can escape to infinity, is called the "event horizon". According to the classical theory, the area of this event horizon can never decrease and will increase as more matter or radiation fall into the black hole (Hawking 1971a, 1972, 1973). This result, which has been called the Second Law of black holes (Bardeen, Carter and Hawking 1973), means that one cannot recover the rest-mass energy of the original body, apart from the fraction (less than 29%) which may be extracted as the rotational energy of the black hole.

The classical theory of general relativity must, however, be regarded as only an approximation to a deeper quantum theory of space-time. In such a theory one would not expect the space-time metric to be defined exactly but to have on a length-scale L an uncertainty or fluctuation of the order of L_o/L where L_o is the Planck length 10^{-33} cm. These quantum fluctuations of the metric will mean that the event horizon will not be defined exactly. It is therefore possible for energy and radiation to tunnel out of the black hole and to escape to infinity, thus causing the hole to appear not completely black. I shall show that this quantum mechanical emission from so-called "black" holes takes place at a steady nonzero rate. As one might expect from the "no-hair" theorems, this rate depends only on the mass M, the angular momentum J and the electric charge Q. More remarkably, the flux and the spectrum of the emitted radiation are exactly what one would get if the black hole were an ordinary body with a temperature of $\frac{\kappa}{2\pi}$ in units such that $G = c = 1$ where κ is the surface gravity of the black hole and is equal to

$$\kappa = \frac{(M^4 - J^2)^{1/2}}{2M(M^2 + (M^4 - J^2)^{1/2})}$$

(Bardeen, Carter and Hawking 1973). Although surprising, this result fits in very nicely with the correspondence between thermodynamics and black hole mechanics that has been pointed out by Bekenstein (1973) and by Bardeen, Carter and Hawking (1973). The starting point for this correspondence is the obvious resemblance between the second law of thermodynamics and the law that, classically at least, the area A of the event horizon can never decrease. There is also an analogy with the first law of thermodynamics in the result that two neighbouring black hole equilibrium states are related by

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

where Ω and Φ are respectively the angular velocity and electrostatic potential of the black hole (Carter 1973). Comparing this to

$$dU = TdS + PdV$$

one sees that, if the area A is associated with entropy, then the surface gravity κ must be associated with temperature. Indeed Bekenstein suggested that some multiple of κ should be regarded as the temperature of the black hole and he postulated a "Generalised Second Law": entropy plus some multiple of area never decreases. However he did not specify the multiple and he did not suggest that black holes could emit radiation as well as absorbing it. In the absence of such emission the Generalised Second Law would be violated by a black hole immersed in black body radiation at a lower temperature: the increase of area caused by absorption of the radiation would be insufficient to counter-balance the loss of entropy down the black hole. This violation is removed if one accepts that black holes emit thermal radiation with a temperature of $\frac{\kappa}{2\pi}$.
The Generalised Second Law then becomes: $S + A/4$ never decreases.

The quantum mechanical emission of radiation by black holes is the direct analogue of the particle creation that occurs in flat space-time in the presence of a deep potential well of some external field such as an electromagnetic field. This process is fairly well understood and has been discussed by a number of authors. One way of visualising it is as follows. In a deep potential well there will be particle states with negative energy with respect to infinity. It is therefore possible to have spontaneous creation of pairs of particles, one particle having positive energy can escape to infinity while the other particle having negative energy remains in the potential well. The situation with black holes is very similar: a black hole is a deep well in the gravitational field inside which there are particle states with negative energy with respect to infinity. The difference between black holes and potential wells in flat space-time is that in the black hole case the region containing negative energy states is separated from infinity by the event horizon. One therefore has to think of the negative energy particle tunnelling into the black hole from the classically forbidden exterior region. Equivalently, one could think of the creation as taking place inside the hole with the positive energy particle escaping through quantum fluctuations of the metric. Because of the necessity to tunnel in or out of the horizon, the rate of particle creation by black holes is relatively low and is governed by the surface gravity since this measures the gravitational field at the horizon and hence how far the particles have to tunnel.

Consider, for simplicity, a quantised scalar Hermitian field ϕ. To calculate the number of scalar particles created by an external potential well one has to solve the field equation for ϕ with the external potential. In order to obtain the correct initial conditions with no scalar particles present, I shall assume that the potential well is not present in the infinite past but that it develops at some later time. One can therefore express the Heisenberg operator in the form:

$$\phi = \sum_i \left\{ f_i a_i + \bar{f}_i a_i^\dagger \right\}$$

where the $\{f_i\}$ are a complete orthonormal family of c-number solutions of the wave equation (with the external potential) which in the infinite past contain only positive frequencies. The operators a_i and $a_i^\dagger$ are position independent and commute with each other apart from $[a_i, a_j^\dagger] = \delta_{ij}$. In the infinite past, when the potential well was absent, a_i and $a_i^\dagger$ have the interpretation of annihilation creation operators respectively. Thus the state $|0_-\rangle$ which contains no scalar particles in the infinite past is defined by

$$a_i \,|0_-\rangle = 0$$

for all i. As the solutions $\{f_i\}$ propagate forwards in time they will be affected by the presence of the potential. Thus to an observer at infinity at late retarded times they will appear to contain negative frequency components. This means that for the observer the operators a_i will not be the annihilation operators and the state $|0_-\rangle$ will not appear to be empty. Let $\{p_i\}$ be a complete orthonormal family of outgoing c-number solutions of the wave equation which are asymptotically positive frequency near infinity at late retarded times. The operator ϕ can be expressed in terms of the $\{p_i\}$ and some additional solutions $\{q_i\}$ which represent states which at late times are localised in the potential well and do not extend to infinity:

$$\phi = \sum_i (p_i b_i + \bar{p}_i b_i^\dagger + q_i c_i + \bar{q}_i c_i^\dagger)$$

The position independent operators b_i and $b_i^\dagger$ are respectively the annihilation and creation operators for an observer at infinity at late retarded times. Each solution p_i can be expressed as a linear combination of solutions $\{f_i\}$ and $\{\bar{f}_i\}$:

$$p_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} \bar{f}_j) \quad .$$

Using this and comparing the two expressions for ϕ , one sees that each operator b_i can be expressed as a linear combination of $\{a_i\}$ and $\{a_i^\dagger\}$:

$$b_i = \sum_j (\bar{\alpha}_{ij} a_j - \bar{\beta}_{ij} a_j^\dagger) \;.\,.$$

For the observer at late retarded times the number operator for the ith outgoing state will be $b_i^{\dagger} b_i$. The expectation value of this in the state $|0_-\rangle$ will be $\sum_j |\beta_{ij}|^2$. Thus to compute the number of particles produced by the potential well one simply has to find the coefficients β_{ij}. In this derivation of particle creation it is essential that the potential well was turned off at early times because otherwise there would have been no mixing of positive and negative frequencies. However the creation should not be thought of as all taking place during the time that the potential well was being turned on. Particle creation is really a global process which cannot be localised. The rate of particle emission by the potential well at late retarded times is independent of exactly how the potential well was turned on.

The above discussion applies to particle production by any external potential. To apply it to the gravitational case one simply has to replace the Minkowski metric η_{ab} by g_{ab} in the wave equation for ϕ, which, for simplicity, I shall take to be massless. I shall consider the field ϕ in an asymptotically flat space-time containing a black hole which, for reasons explained above, is assumed not to have existed for all time but to have formed from the collapse of some body. One can describe the initial positive frequency solutions $\{f_i\}$ as solutions which are positive frequency along the generators of past null infinity $\mathcal{J}^-$ (Penrose 1964, Hawking 1973, Hawking and Ellis 1973). The outgoing solutions $\{p_i\}$ are positive frequency along the generators of future null infinity $\mathcal{J}^+$ and are zero on the event horizon. The solutions $\{q_i\}$ are zero on $\mathcal{J}^+$. It is not necessary to define positive frequency for them.

In order to calculate the particle production it is more convenient to use solutions $\{f_{\omega'}\}$ and $\{p_\omega\}$ with continuum normalisation rather than finite normalisation. Consider such a solution p_ω propagating backwards from $\mathcal{J}^+$ (fig. 2). A part $p_\omega^{(1)}$ of the solution will be scattered by the curvature of the stationary black hole solution and will end up on $\mathcal{J}^-$ with the same frequency ω that it had initially. This will give rise to a $\delta(\omega - \omega')$ behaviour in the coefficients $\alpha_{\omega\omega'}$. More interesting effects arise from the other part $p^{(2)}_{\omega}$ of the solution which enters the collapsing body, passes through the centre and travels out to $\mathcal{J}^-$. Because the surfaces of constant retarded time get all squashed up near the horizon (fig. 2), $p^{(2)}_{\omega}$ gets a very large blue shift. It therefore propagates by geometric optics through the collapsing body and out on to $\mathcal{J}^-$. On each generator of $\mathcal{J}^-$ it will have the asymptotic form

$$\begin{array}{lll} C \exp(-\frac{\omega}{\kappa} \log(v_0 - v)) & \text{for} & v < v_0 \\ 0 & \text{for} & v > v_0 \end{array}$$

where C is a constant, v is advanced time and v_0 is the latest advanced time at which a particle could leave $\mathcal{J}^-$, pass through the centre of the collapsing body and escape to $\mathcal{J}^+$ before being trapped inside the event horizon. Taking Fourier transforms one finds that the coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ have the asymptotic form for large ω'

$$\alpha^{(2)}_{\omega\omega'} = D(\omega')^{-\frac{1}{2} + i\frac{\omega}{\kappa}}$$

$$\beta^{(2)}_{\omega\omega'} = -iD(-\omega')^{-\frac{1}{2} + i\frac{\omega}{\kappa}}$$

There is a logarithmetic singularity in these expressions at $\omega' = 0$. Since $p^{(2)}_{\omega}$ is zero for large advanced times v, its Fourier transform will be analytic in the upper half ω' plane. This means that to get the correct ratio of $\alpha^{(2)}_{\omega\omega'}$ to $\beta^{(2)}_{\omega\omega'}$ one has to continue $\alpha^{(2)}_{\omega\omega'}$ anticlockwise round the singularity and then replace ω' by $-\omega'$. Thus for large ω'

$$|\alpha^{(2)}_{\omega\omega'}| = \exp(\pi\omega\kappa^{-1})|\beta^{(2)}_{\omega\omega'}| .$$

The number of particles created and emitted to infinity in the frequency range ω to $\omega + d\omega$ is

$$d\omega \int_0^\infty |\beta_{\omega\omega'}| \, d\omega' .$$

By the asymptotic expression above this is infinite. To see that this infinite total number of particles emitted corresponds to a steady emission at the thermal rate consider a finite normalisation outgoing wave packet mode

$$\int_0^\infty \psi_\omega p_\omega \, d\omega$$

where

$$\int_0^\infty \psi_\omega \bar{\psi}_\omega \, d\omega = 1$$

The number of particles that will be emitted in this wave packet mode will be

$$\iint \psi_\omega \beta_{\omega\omega'} \, d\omega \, d\omega' .$$

The fraction Γ of the wave packet that enters the collapsing body will be

$$\iint |\psi_\omega|^2 \left(|\alpha^{(2)}_{\omega\omega'}|^2 - |\beta^{(2)}_{\omega\omega'}|^2 \right) d\omega \, d\omega' .$$

It follows from this that if the wave packet is sharply peaked around a frequency of ω , the number of particles emitted in the wave packet mode will be

$$\Gamma \left(\exp(2\pi\omega\kappa^{-1}) - 1 \right)^{-1} .$$

For a wave packet at late retarded time, the fraction Γ that enters the collapsing body is almost the same as the fraction that would have crossed the past event horizon of the analytically extended final stationary black hole solution. This in turn is equal to the fraction of the wave packet that would have been absorbed by the black hole had the wave packet come from $\mathscr{I}^-$. This is exactly the relation between emission and absorption cross sections for a body of temperature $\frac{\kappa}{2\pi}$. Similar results hold for other fields of integer spin and with rest mass. For Fermion fields the results are again similar except that the thermal factor is

$$\left(\exp(2\pi\omega\kappa^{-1}) + 1 \right)^{-1}$$

The temperature of a black hole is of the order of $10^{26}M^{-1}\ {}^{o}K$ where M is the mass in grams. The thermal emission means that black holes will radiate all their rest mass in time of the order of $10^{-28}M^{3}$ secs. For black holes formed by collapsing stars $M \sim 10^{33}$ grams so the temperature is very low and the life time is much longer than the age of the universe. However there might be much smaller black holes which were formed by fluctuations in the early universe (Hawking 1971b). Any such black hole of less than 10^{15} grams would have evaporated by now. Even if there aren't any small black holes, the fact that black holes are not completely black is conceptually very important because it means that they are not completely cut off from the rest of the universe.

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