THE CREATION AND ANNIHILATION OF MATTER BY A GRAVITATIONAL FIELD

by

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Summary

It is shown that in classical general relativity, if space-time is nonempty at one time, it will be nonempty at all times provided that the energy momentum tensor of the matter satisfies a physically reasonable condition. The apparent contradiction with the quantum predictions for the creation and annihilation of matter particles by gravitons is discussed and is shown to arise from the lack of a good energy momentum operator for the matter which obeys the covariant conservation equation.

1 Introduction

In the general theory of relativity it is well known that the equations $T^{ab}{}_{;b} = 0$, which express the local conservation of energy and momentum, cannot be integrated to give conservation laws over a region. This is because the tensor T^{ab} represents the energy-momentum only of matter fields and not of the gravitational field. It is a matter of common experience that the energy of a system is not conserved unless one also takes into account its gravitational energy. In Newtonian theory the concept of gravitational energy is well defined but in general relativity this is unfortunately not the case for arbitrary fields. However for a bounded system in an asymptotically flat space-time one can define a total energy or mass which represents the energy of both the matter and the gravitational field and which decreases at a rate which can be interpreted as the rate at which energy is carried away to infinity by gravitational radiation. (Bondi, van der Burg and Metzner 1962, Newman and Unti 1962, Penrose 1963, Price and Thorne 1969). The question then arises: could the system radiate away all its mass as gravitational waves and leave just empty space? It will be shown that it could not in the classical theory if T^{ab} satisfies a physically reasonable condition. That is to say a space-time which is non-empty at one time must be non-empty at all times and conversely that one which is empty at one time must be empty at all times. This result depends only on the equations $T^{ab}{}_{;b} = 0$ and not on the field equations.

This result would seem to be in contradiction with the non zero cross-sections which have been calculated for such processes as the annihilation of a pair of particles into gravitons (see, for example, De Witt

1968). The reason for this discrepancy between classical and quantum theories are discussed in Section 3. It seems to be related to the difficulty of defining a local energy-momentum operator for the matter fields in a curved space-time.

2 Classical Theory

The energy-momentum tensor will be said to satisfy the dominant energy condition if for every observer the local energy density $T_{ab}U^aU^b$ is non-negative and the local energy flow vector $T_{ab}U^a$ is non-spacelike where U^a is the velocity vector of the observer ($U^a U_a > 0$). The first of these would seem to be required by local quantum mechanics and the second by causality. They are satisfied by all known forms of matter. With a bit of algebra the dominant energy can be shown to be equivalent to the requirement that in any orthonormal tetrad the energy component T_{oo} should be greater than or equal to the absolute values of the other components of the energy-momentum tensor i.e.

$$T_{oo} \geq |T_{ab}| \quad \text{for each } a, b.$$

For a fluid with energy density μ and principal pressures p_i ($i = 1, 2, 3$) this will hold if and only if $\mu \geqslant p_i$. This is a very reasonable requirement since if the pressure were to exceed the energy density, there would be sound waves which travelled faster than light.

We shall show that if the dominant energy condition holds, a region which is empty initially remains empty provided no matter flows in from outside. To be more precise we shall show that T_{ab} vanishes in a compact region F whose boundary ∂F consists of a part $(\partial F)_1$ on which T_{ab} vanishes and a part $(\partial F)_2$ whose outward normal is future directed and timelike or null. We shall assume that there is a function t whose gradient is everywhere future directed and timelike. Such a function will exist provided space-time is not on the verge of violating causality (Hawking 1968).

Let $H(t')$ denote the spacelike surface $t = t'$ and let $F(t')$ denote the part of F to the past of $H(t')$. Consider the volume integral

$$I(t) = \int_{F(t)} (T^{ab} t_{;a})_{;b} \, dv = \int_{F(t)} T^{ab} t_{;ab} \, dv$$

where $dv = \sqrt{-g}\, d^4x$ is the invariant volume element. By Gauss's theorem this can be transformed into an integral over the boundary of $F(t)$,

$$I(t) = \int_{\partial F(t)} T^{ab} \, t_{;a} \, dS_b$$

The boundary of $F(t)$ will consist of $F(t) \cap (\partial F)_1$, $F(t) \cap (\partial F)_2$ and $F \cap H(t)$. Since T_{ab} is zero on $(\partial F)_1$,

$$I(t) = \int_{F(t) \cap (\partial F)_2} + \int_{F \cap H(t)} .$$

By the dominant energy condition the first term on the right is non-negative since the outward normal to $(\partial F)_2$ is future directed. Thus

$$I(t) = \int_{F(t)} T^{ab} t_{;ab} \, dv \geqslant J(t) = \int_{F \cap H(t)} T^{ab} \, t_{;a} \, dS_b \ .$$

Since F is compact there will be some upper bound to the components of $t_{;ab}$ in any orthonormal tetrad whose timelike vector is in the direction of $t_{;a}$. Thus, by the dominant energy condition there will be some $C > 0$ such that on F

$$T^{ab} t_{;ab} \leqslant C \, T^{ab} \, t_{;a} t_{;b}$$

The integral over $F(t)$ can be decomposed into an integral over the surfaces $H(t')$ followed by an integral with respect to t':

$$I(t) \leqslant C \int^{t} \left(\int_{F \cap H(t')} T^{ab}\, t_{;a}\, dS_b \right) dt'$$

Therefore $\frac{d}{dt} J(t) \leqslant C\, J(t)$.

But for sufficiently early values of t, H(t) will not intersect F and so J(t) will vanish. Thus J(t) will vanish for all t which implies that T^{ab} vanishes on F.

In other words, if space-time is empty at one time, then it will remain empty providing that matter does not come in from infinity. Conversely, a bounded system which is present at one time cannot fail to be present at all other times.

3 Quantum Theory

To try to explain the apparent contradiction between the result of the previous section and the nonzero cross-sections which have been calculated for the creation or annihilation of particles by gravitons we shall consider the case of a Hermitian scaler field ϕ of mass m which obeys the covariant Klein-Gordon equation

$$g^{ab}\phi_{;ab} + m^2\phi = 0.$$

As the field equations were not used in previous section, we can take the metric to be a given external (unquantised) field unrelated to any matter content. For simplicity we shall consider a space-time consisting of initial and final flat space regions M_1 and M_3 separated by a region M_2 in which the curvature is nonzero. In M_1 and M_3 the operator ϕ behaves as that of a free field and may be given the standard quantum field theory interpretation. We may decompose ϕ into ϕ_1^+ which has only positive frequencies on M_1 and ϕ_1^- which has only negative frequencies on M_1 i.e.

$$\phi(x) = \phi_1^{+}(x) + \phi_1^{-}(x)$$

$$= \sum_{\alpha} \left(f_{1\alpha}(x)\, a_{1\alpha} + \bar{f}_{1\alpha}(x)\, a^{*}_{1\alpha} \right)$$

where the $a_{1\alpha}$ are operators independent of position and the $f_{1\alpha}$ are a complete orthonormal set of complex functions which satisfy the covariant Klein-Gordon equation and contain only positive frequencies on M_1. Similarly ϕ may be decomposed into ϕ_3^{+} and ϕ_3^{-}, its positive and negative frequency parts on M_3. However because of the intervening curvature in the region M_2, $\phi_3^{+}(x)$ will not in general be equal to $\phi_1^{+}(x)$. In other words, a solution $f_{1\alpha}$ of the covariant Klein-Gordon equation which has only positive frequency components on M_1 may have a negative frequency component on M_3. It is this intermixing of the positive and negative frequencies which is responsible for the creation or annihilation of particles by the gravitational field. For in M_1, $\phi_1^{+}(x)$ and $\phi_1^{-}(x)$ may be interpreted as the operators which respectively annihilate and create a particle at the point x. Thus the condition on the state-vector $|\ \rangle$ that there should be no particles in M_1 is not that $\phi(x)|\ \rangle = 0$ but that $\phi_1^{+}(x)|\ \rangle = 0$ for all x in M_1. By the covariant Klein-Gordon equation $\phi_1^{+}(x)|\ \rangle$ will then also be zero for x in M_2 or M_3. However the particle annihilation operator in M_3 is $\phi_3^{+}(x)$ and because of the frequency mixing, $\phi_3^{+}(x)|\ \rangle$ may be nonzero corresponding to a nonzero probability of finding a particle at the point x in M_3. For any reasonable gravitational field this probability will be very low. Parker (1968) has calculated that the rate of production of π^0 mesons by the gravitational field of the universe is about one particle a second in the whole universe.

The energy-momentum tensor for a classical scalar field ϕ satisfying the covariant Klein-Gordon is

$$T_{ab} = \phi_{;a}\phi_{;b} + \tfrac{1}{2}g_{ab}(m^2\phi^2 - \phi_{;c}\phi_{;d}g^{cd}).$$

This obeys $T^{ab}{}_{;b} = 0$ and satisfies the dominant energy condition. There is difficulty however in defining an energy-momentum operator in quantum theory for if in the above expression one regards the ϕ's as operators, one obtains an operator whose expection value even in flat, empty space is infinite. This is because there are negative frequency parts (creation operators) of ϕ standing to the right of positive frequency parts (annihilation operators). In flat space-time one way of overcoming this difficulty is to adopt normal ordering for products of ϕ's i.e. decompose the ϕ's into positive and negative frequency parts and rearrange the terms so that the negative frequencies are to the left and the positive to the right. However this will not work in curved space-time since, as we have seen, this splitting into positive and negative frequency parts is not possible in general. Another possible way round the difficulty would be to try to define the energy momentum operator as the limit of a nonlocal operator in which the two ϕ operators are referred to different points and the vacuum expectation value is subtracted out. However this operator does not satisfy the covariant conservations $T^{ab}{}_{;b} = 0$ in a general space-time. In fact there is no operator which satisfies these equations and has the right properties to be interpreted as the energy-momentum operator. One may ask: How then is the energy-momentum tensor to be defined which appears on the right of the Einstein equations? The answer is that in the classical correspondence limit the energy-momentum tensor may be taken to have the classical form where the function $\phi(x)$ is the expectation value $\langle \phi(x) \rangle$ of the operator. Note that this expectation value may vanish even when there are particles present (in flat space it vanishes for any state with a definite number particles). However the classical correspondence limit is made for a state which does not have an exact number of particles but is a superposition of a number of states:

$$| \rangle = \sum_{n_1, n_2, \cdots}' C_{n_1 n_2 \cdots n_k} |n_1, n_2, \cdots, n_k\rangle$$

where $|n_1, n_2, \cdots, n_k\rangle$ denotes a state which has initially n_1 particles in the state described by f_1 and so on. If the constants $C_{n_1 n_2 \cdots n_k}$ vary slowly with n_1, n_2 . . . and the slight frequency mixing is neglected

$$\langle \phi(x) \rangle \simeq \sum_{n_1, n_2, \cdots}' C^2_{n_1 n_2 \cdots n_k} \sum_{\alpha=1}^{k} \sqrt{n_\alpha} \, (f_\alpha + \bar{f}_\alpha) \quad .$$

The energy-momentum tensor constructed from this expectation value satisfies the covariant conservation equation and gives a reasonable representation of the energy-momentum of the particles present ab initio. It does not, however, represent the creation or annihilation of particles by the gravitational field as it vanishes everywhere in the example given above. In any normal situation this creation or annihilation may be neglected to a high order of accuracy.

Conclusion

In classical theory physically reasonable matter cannot be created or annihilated by a gravitational field. In quantum theory creation and annihilation are possible but in any normal field the rate would be utterly negligible. It is only near a space-time singularity that it might become significant.

References

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Biographical Sketch

I was born in 1942 in Oxford, England. I attended St. Albans School and Oxford University, where I obtained a B.A. in Physics in 1962. I then came to Cambridge to do research in general relativity and cosmology. I became a Fellow of Gonville and Caius College in 1965 and took my Ph.D. in 1966. Since last year I have held a post at the Institute of Theoretical Astronomy.

I am married with a son aged about two.

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