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THE GRAVITATIONAL COLLAPSE OF THE UNIVERSE

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It is shown that if gravity is always attractive and if the Universe obeys the Copernican Principle that it is spatially homogeneous, then there must be a singularity where the density is infinite either in the future or in the past. If, as seems to be the case, the singularity is in the past, this implies that the Universe had a beginning.

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Gravitational collapse of massive objects has attracted considerable attention recently as it has been suggested as an energy source for Quasi-Stellar sources (1). This interest is heightened because it seems that the collapse will not stop but will contin until a singularity is reached at which the density is infinite (2). However, an observer outside the massive body can never see the singularity on account of the event-horizon at the Schwarzschild radius. It has been argued therefore that he may ignore its existence. This escapist view will not serve, however, when we consider the gravitational collapse of the whole universe, for, in this case, the observer is inside and so must be intimately concerned.

The Sign of Gravity

The immediately obvious fact about gravity is that it seems to be always attractive. Indeed it is only for this reason that we notice it at all. For gravity is the weakest force known and it is only because the field due to every particle in the earth has the same sign that there is a significant field at the surface of the earth. It seems reasonable to assume therefore that gravity always appears attractive to any observer. A mathematical statement of this would be that the relative accleration of two neighbouring geodesics is always towards each other.

Another way of looking at this is to consider the Einstein field equations:

$$R_{ab} - \frac{1}{2} g_{ab} R = K T_{ab}$$

where R_{ab} is the Ricci-tensor and T_{ab} the energy-momentum tensor of the matter fields. Call $T = T_a^{\ a} = -\frac{1}{K} R$ the rest-mass density of the matter fields and $E = T_{ab} V^a V^b$ the energy-density in the rest frame of an observer with 4 - velocity V^a. Then if $E \geqslant \frac{1}{2} T > 0$ for any time-like V^a, $R_{ab} V^a V^b \geqslant 0$ and gravity will always appear attractive in the sense used above. Note that this will not necessarily be the case if the field equations contain a λ term or if there is a 'C' field of the type proposed by Hoyle (3). There is however a grave objection to the existence of a field with a negative energy density like the 'C'-field. For, in a quantum theory, there would be nothing to prevent the creation, in a given 4 - volume of space, of an infinite number of quanta of the 'C'- field and a corresponding infinity of quanta of ordinary matter. This is connected with an indeterminacy of the 'C'- field equations which has been remarked by Raychaudhuri (4). We will assume therefore that there are no fields with negative energy densities and so gravity is always attractive.

The Copernican Principle

In the Ptolomeic model of the universe, Man, who considered himself the most important being in the Cosmos, placed himself at the centre. Since the time of Copernicus, however, we have had to give up this vanity and we are now so modest that we assume that all parts of the universe at the same time are essentially the same. We call this the Copernican Principle. A mathematical statement of this would be that the universe has a 3 - parameter group of motions whose surface of transitivity is a space-like 3 - surface. Note that we do not demand that the universe be isotropic. If we did, we would exclude rotation of the matter and it has been shown by Robertson and Walker (5) and Raychaudhuri (6) that in this case there must be a singularity of space-time. However it will be shown that, provided gravity is always attractive and the Copernican

Principle holds, even the existence of rotation cannot prevent the occurrence of a singularity.

Singularity

Consider a space-like hypersurface H of transivity of the group. It must be a surface of constant density μ. Therefore $\mu_{;a}$ will be in the direction of the normal to the surface; also $\mu_{;a}\mu^{;a} = f > 0$ where f is constant on the 3 - surface.

Therefore $f = f(\mu)$

Let $$V^a = \frac{e(\mu_{;a})}{\sqrt{f}}\mu^{;a}$$

where $e(\mu_{;a})$ = +1 if $\mu_{;a}$ is in the + time direction
-1 if $\mu_{;a}$ is in the -time direction

Then
$$V_a V^a = 1$$
$$V_{a;b} V^b = 0$$
$$V_{[a;b]} = 0$$

Therefore V_a is a congruence of time-like irrotational geodesic unit vectors.

if
$$\theta = V_a{}^{;a}$$
$$\sigma_{ab} = V_{(a;b)} - \tfrac{1}{3}\theta(g_{ab} - V_a V_b)$$
then
$$\theta_{;b} V^b = -\tfrac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}V^a V^b$$ (Raychaudhuri's equation)

but
$$\sigma_{ab}\sigma^{ab} \geqslant 0$$
$$R_{ab}V^a V^b > 0$$ (by the assumption that gravity is attractive)
$$\therefore \theta_{;b} V^b < -\tfrac{1}{3}\theta^2$$

if $\theta \leqslant 0$ then θ will become infinite in finite proper time if $\theta > 0$

let $V'_a = -V_a$

$\theta' = -\theta$

$\theta'_{;b} V'^{b} < -\frac{1}{3} \theta'^2 \qquad \theta' < 0$

$\therefore \theta'$ will become infinite in finite proper time.

If we assume that the surfaces μ = const. remain space-like, then this means that the convergence of their unit normals becomes infinite. This implies that they must degenerate into, at the most, a 2 - surface; call this C . Consider the fluid flow vector u_a . This is uniquely defined by the geometry as the timelike eigenvector of the Einstein tensor:

$$u^a(R_{ab} - \tfrac{1}{2} g_{ab} R) = -\mu u_b$$

Now consider all fluid flow lines, F crossing H. If they do not all intersect C, let be the subset of F that intersect C. Let M be the subset of H intersected by L . Since C is uniquely defined by the geometry, L and M will also be uniquely defined. But, by the original postulate, H was a completely homogeneous surface, therefore it can have no uniquely defined subset. Hence all the fluid flow lines F must intersect C and the density will be infinite there.

The above was on the assumption that the surfaces μ = const. remain space-like. They may however become light-like ($f = 0, \mu_a \neq 0$). It will be shown that in this case too there must be singularities.

Consider the surface = const. on which

$$\mu_{;a}\mu^{;a} = f = 0 \qquad \mu_a \neq 0$$

then $\mu_{;ab}\mu^{;a} = \mu_{;ba}\mu^{;a} = \frac{f'}{2}\mu_{;b}$

Introduce a geodesic null congruence L_a throughout space

$$L_a L^a = 0$$

$$L_{a;b} L^b = 0$$

also take L_a to be proportional to $\mu_{;a}$ on the surface $f = 0$

$$L_a = \alpha \mu_{;a}$$

Introduce null vectors $n_a, m_a, \bar{m}_a$ where:

$$L_a n^a = 1, \qquad m_a \bar{m}^a = -1$$
$$L_a m^a = n_a m^a = 0$$

then one may write:

$$\begin{aligned} L_{a;b} = q L_a L_b &+ \rho(m_a \bar{m}_b + m_b \bar{m}_a) + \bar{\sigma} m_a m_b \\ &+ \sigma \bar{m}_a \bar{m}_b + s L_a m_b + \bar{s} L_a \bar{m}_b \\ &+ t L_b \bar{m}_a + \bar{t} L_b m_a \end{aligned}$$

then $L_a{}^{;a} = -2\rho$

$$\begin{aligned} (L_a{}^{;a})_{;b} L^b &= L_{a;b}{}^{a} L^b + R^c{}_a{}^a{}_b L_b L^c \\ &= -2\rho^2 - 2\sigma\bar{\sigma} - R_{bc} L^c L^b \end{aligned}$$

But $R_{cb} L^c L^b > 0$ by the assumption that gravity is attractive and the space is not empty.

therefore $L_a{}^{;a}$ will become infinite within a finite affine distance.

The only freedom we have in the choice of L_a on the surface $f = 0$ is:

$$L'_a = \beta L_a$$

where $\beta_{;b} L^b = 0$

i.e. β is constant along a null ray

$$L'_a{}^{;a} = \beta L_a{}^{;a}$$

Thus $L'_a{}^{;a}$ will be infinite when $L_a{}^{;a}$ is infinite. Therefore the 2 - surface on the 3 - surface $f = 0$, on which $L_a{}^{;a}$ is infinite will be uniquely defined. We may then apply the arguments used before to show that the density will be infinite there.

Of course the singularity could be in the past not in the future. In this case we would speak of the universe expanding away from a collapsed state rather than collapsing towards a state of infinite density. Indeed this would seem to be true of our universe. The question whether our universe will fall back again or whether it is expanding fast enough to avoid this is one that can only be settled by observation. So far the results are inconclusive.

Since space-time cannot be extended through a singularity, a singularity in the past would mean that the universe had a 'beginning'. In particular it appears that an oscillating universe is incompatible with attractive gravity and the Copernican Principle.

The proof of the occurrence of a singularity given above has depended on the existence of a group of motions. Now, of course the universe has no exact group since there are local irregularities such as stars and galaxies, However, provided that the universe is spatially homogeneous on a sufficiently large scale it is possible to extend the pro and show that a singularity still occurs.

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I was born in 1942 in Oxford, England and was educated at St. Albans School. In 1959 I went up to Oxford University where I took a degree in Physics. While engaged upon vacation work at the Royal Greenwich Observatory I became interested in Cosmology and consequently came to Cambridge to do research under Dr. Sciama. I complete my Ph.D this summer and shall then take up a Research Fellowship at Gonville and Caius College, Cambridge.