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The Interaction of Gravity with Quantized Fields

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ABSTRACT

A rapidly growing area of gravity research is the study of particle creation by gravitational fields, an effect predicted by quantum field theory. To calculate the influence of the created particles on the gravitational field producing them, it is necessary to find the physically effective renormalized energy-momentum tensor. We discuss an analysis of the concept of physical particle states which seems to lead in a natural manner to the renormalized tensor. Included is a demonstration that our definition of particle states yields physically correct results when applied to the particular Kasner universe which reduces to Minkowski space.

Among the most interesting and important topics under current study in the theory of gravitation are the various types of quantum phenomena which occur in the early universe and in gravitational collapse, because of the presence of strong or time-dependent gravitational fields. In many applications it is sufficient to treat the gravitational field classically, while describing the matter and radiation by means of quantized fields. Such investigations are usually carried out in two general stages.

First, the gravitational field is taken as given (for example, one of the known solutions of the classical Einstein equations), and the quantum phenomena growing out of it are studied, ignoring their influence on the spacetime which produces them. At this stage one already finds the important phenomenon of the creation of elementary particles (e.g., neutrinos, photons, electrons, pions) by the gravitational field. This particle creation and its classical analogues (backscattering and superradiance) have been discussed in cosmological metrics [1-16], and also in black hole geometries and for gravitationally collapsing objects [17-23].

The second stage of development involves the influence, or reaction, of the particle creation on the gravitational field. In this case the expectation value of the energy-momentum tensor of the quantized particle fields acts as the source of the Einstein gravitational field, and one solves for the metric, thus taking into account the particle creation. Such calculations have been carried out so far primarily in the cosmological context [24-25], with the aims of explaining the origin of the observed isotropy and homogeneity of the universe and of investigating the effects on the cosmological singularity. Of prime concern in this type of problem is the definition of the physically relevant energy-momentum tensor, since the formal expression given by quantum field theory has infinite expectation values. Techniques of regularization and renormalization for dealing with that problem are being developed [26-27, 10, 28-30]. For homogeneous, time-dependent spaces which are not necessarily asymptotically flat, such as one encounters in cosmology, the definition of the physical energy-momentum tensor is naturally closely related to the problem of finding the physically relevant generalization of the concept of single-particle states of definite energy and momentum (i.e., positive- and negative-frequency solutions of the particle field equations). The rest of this essay will be devoted to a discussion of a physically motivated generalization of the decomposition into positive- and negative-frequency solutions, and to the presentation of a new result which bears on the question of the general covariance of this decomposition.

To write down a canonical quantum field theory against a curved background metric is straightforward, but to interpret it physically in terms of particles is a subtle problem [see 3, 6, 12, 31-33]. We have been considering this question for time-dependent but (at least so far) spatially homogeneous spacetimes, and have obtained encouraging results. One notes first that if a time-dependent metric is static during certain intervals, then the number of particles present is well defined at those times and is an adiabatic invariant. More precisely, when one considers a family of similar metrics (initially and finally static), differing in their rate of time variation, one finds that the density of particles created from the vacuum vanishes faster than any power of that rate as it tends to zero. This suggests that particle observables can and should be defined in a general time-varying situation (not necessarily static at any time) so that the particle number is an adiabatic invariant to the greatest possible extent [3, 6]. The particle concept in a time-varying metric can in general only be approximate, since, in analogy to the time-energy uncertainty relation, the very fact that the particle number is changing makes its measurement to arbitrary accuracy impossible [6, Sec. E].

The adiabatic construction of particle states is implemented [3, 28, 30] by solving the field equation in the adiabatic (slow) limit in a generalized WKB approximation. In the static case the wave function of a particle must be a positive-frequency solution: the characteristic form of its time dependence is $(2\omega)^{-\frac{1}{2}} e^{-i\omega t}$. When the metric is changing, we define a solution to have positive frequency if it is well approximated at all times in the adiabatic limit by a superposition of functions of the form

$$(2W)^{-\frac{1}{2}} e^{-i\int^{t} W(t')dt'} ,$$

as opposed to those with +i in the exponential. Here t is an appropriately rescaled time coordinate, and W(t) depends only on the metric and its time derivatives at t. W can be chosen so that the approximation is valid to an arbitrarily high order in the rate of time dependence [34]. The fact that the solution is not precisely of this form, but rather develops a component of the opposite frequency, is the mathematical expression of the phenomenon of particle creation. Once the concept of "positive frequency" is established, the meaning of "particles" and the "vacuum state" follows from the usual development of quantum field theory.

The adiabatic definition of the vacuum state is leading toward a convincing definition of the effective energy-momentum tensor. We expand the infinite formal expression for the tensor in a power series in a parameter representing the curvature of the spacetime — in particular, this entails going to the adiabatic limit as regards the time dependence. The divergent leading terms of this series can, after some manipulation, be shown to be formally proportional to tensors describing the geometry of the spacetime: the metric tensor $g_{\mu\nu} = -4|g|^{-\frac{1}{2}}\delta[|g|^{\frac{1}{2}}]/\delta g^{\mu\nu}$, the Einstein tensor $G_{\mu\nu} = 2|g|^{-\frac{1}{2}}\delta[|g|^{\frac{1}{2}}R]/\delta g^{\mu\nu}$, and two quadratic tensors,

$$^{(1)}H_{\mu\nu} = 2|g|^{-\frac{1}{2}}\delta[|g|^{\frac{1}{2}}R^2]/\delta g^{\mu\nu}$$

and

$$^{(2)}H_{\mu\nu} = 2|g|^{-\frac{1}{2}}\delta[|g|^{\frac{1}{2}}R_{\alpha\beta}R^{\alpha\beta}]/\delta g^{\mu\nu} \ .$$

(Here we have emphasized the covariant nature of the quantities by writing them as variational derivatives of the terms in the quadratic scalar Lagrangian of a generalized gravitation theory; $R_{\alpha\beta}$ is the Ricci tensor and R the curvature scalar.) The gravitational effect of these terms is to renormalize the effective physical values of the cosmological constant Λ, the gravitational constant G, and the coupling constants, γ_1 and γ_2, of quadratic terms which must be included in the gravitational field equation.

The remainder of the series is the finite, physical energy-momentum tensor $\langle T_{\mu\nu}\rangle$ which acts as the source in the renormalized field equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \gamma_1 \left({}^{(1)}H_{\mu\nu} + \gamma_2 \; {}^{(2)}H_{\mu\nu} \right) = -8\pi G \langle T_{\mu\nu} \rangle \; .$$

Renormalization in the context of classical gravity and quantum matter has been studied earlier by Utiyama and DeWitt in asymptotically Minkowskian spacetimes [27]. (See also the interesting suggestion of Sakharov [35].) But the adiabatic analysis of the particle concept seems to be an essential ingredient in carrying it out explicitly for a cosmological metric.

The natural appearance of geometric tensors in the asymptotic expansion of the energy-momentum tensor strongly suggests that the adiabatic construction of positive- and negative-frequency solutions is a special case of a more general, manifestly covariant, procedure. Further support for this viewpoint is given by the following argument, which shows that in a particular model admitting two coordinate systems in which the metric is spatially homogeneous, the corresponding definitions of particles are physically equivalent.

In Minkowski space with coordinates (y^0, y^1, y^2, y^3), introduce new coordinates (t, x^1, x^2, x^3) by

$$y^0 = t \cosh x^1, \quad y^1 = t \sinh x^1, \quad y^2 = x^2, \quad y^3 = x^3 \; .$$

The metric in the region where $y^0 > |y^1|$ takes the form

$$d\tau^2 = dt^2 - t^2 (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \; ,$$

a degenerate special case of the Kasner solution [36] of the vacuum Einstein equations, which describes an anisotropically changing, spatially homogeneous universe. We shall show that the adiabatic definition of positive- and negative-frequency solutions in Kasner coordinates is consistent with the standard definition in Minkowski coordinates. The Klein-Gordon equation of a scalar field in Kasner coordinates is

$$\frac{d^2\phi}{dt^2} + \frac{1}{t}\frac{d\phi}{dt} - \frac{1}{t^2}\frac{d^2\phi}{d(x^1)^2} - \frac{d^2\phi}{d(x^2)^2} - \frac{d^2\phi}{d(x^3)^2} + m^2\phi = 0 .$$

Its solutions can be expressed as linear superpositions of the elementary solutions

$$\phi^{\pm}_{\vec{k}} = H^{(j)}_{ik_1}\left(\sqrt{k_2{}^2 + k_3{}^2 + m^2}\; t\right) e^{i\vec{k}\cdot\vec{x}} ,$$

where the $H^{(j)}_{ik_1}$ ($j = 2$ for ϕ^+, $j = 1$ for ϕ^-) are Hankel functions of imaginary index, ik_1. The asymptotic behavior of these functions when their arguments are large [37, p. 962] is

$$\phi^{\pm}_{\vec{k}} \sim \text{const} \times t^{-1/2}(k_2{}^2 + k_3{}^2 + m^2)^{-1/4}\, e^{\mp i\sqrt{k_2{}^2+k_3{}^2+m^2}\; t}\, e^{i\vec{k}\cdot\vec{x}} .$$

It follows that $\phi^{+}_{\vec{k}}$ is a positive-frequency function in the Kasner coordinates by our previously discussed definition. On the other hand, using an integral representation for the Hankel function [37, p. 955] we can express $\phi^{+}_{\vec{k}}$ as a superposition of positive-frequency plane waves in Minkowski space:

$$\phi^{+}_{\vec{k}} = \frac{i}{\pi} e^{-\pi k_1/2} \int_{-\infty}^{\infty} dz\, e^{-i\sqrt{\vec{p}^{\,2}+m^2}\; y^0}\, e^{i\vec{p}\cdot\vec{y}}\, e^{-ik_1 z} ,$$

where $p_1 = -\sqrt{k_2{}^2+k_3{}^2+m^2}\,\sinh z$, $p_2 = k_2$, $p_3 = k_3$. The absence of negative frequencies ($e^{+i\omega y^0}$) in this expression shows that the adiabatic definition of particles in the degenerate Kasner universe agrees with the familiar notion of particles in Minkowski space. In the corresponding quantum field theories the vacuum states can be chosen to be the same, and the theories will be unitarily equivalent.

Our definition of particles has thus passed a stringent test. Results such as these lead us to believe that the adiabatic particle concept and the corresponding renormalization of the energy-momentum tensor are physically correct and will find applications in various studies of the interaction of quantized matter with classical gravitational fields.

REFERENCES

1. E. Schrödinger, Physica 6, 899 (1939).
2. L. Parker, Phys. Rev. D5, 2905 (1972).
3. L. E. Parker, Ph.D. thesis, Harvard University, 1966. (Now available from Xerox University Microfilms, Ann Arbor, Michigan.)
4. R. U. Sexl and H. K. Urbantke, Acta Phys. Austriaca 26, 339 (1967); Phys. Rev. 179, 1247 (1969).
5. L. Parker, Phys. Rev. Lett. 21, 562 (1968).
6. L. Parker, Phys. Rev. 183, 1057 (1969).
7. L. Parker, Phys. Rev. D 3, 346 (1971).
8. A. A. Grib and S. G. Mamaev, Yad. Fiz. 10, 1276 (1969) [Sov. J. Nucl. Phys. 10, 722 (1970)]; Yad. Fiz. 14, 800 (1971) [Sov. J. Nucl. Phys. 14, 450 (1972)].
9. Ya. B. Zeldovich, ZhETF Pis. Red. 12, 443 (1970) [JETP Lett. 12, 307 (1970)].
10. Ya. B. Zeldovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys.-JETP 34, 1159 (1972)].
11. L. Parker, Phys. Rev. Lett. 28, 705 (1972).
12. S. A. Fulling, Ph.D. dissertation, Princeton University, 1972.
13. B. L. Hu, Ph.D. dissertation, Princeton University, 1972; "Scalar Waves in the Mixmaster Universe. II. Particle Creation" (preprint, Stanford University, 1973).
14. B. L. Hu, S. A. Fulling, and L. Parker, Phys. Rev. D 8, 2377 (1973).
15. B. Mashhoon, Phys. Rev. D 8, 4297 (1973).
16. B. K. Berger, Ann. Phys. (N.Y.), to appear.
17. C. W. Misner, unpublished.

18. W. Press and S. Teukolsky, Nature 238, 211 (1972).

19. J. Bekenstein, Phys. Rev. D 7, 949 (1973).

20. A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 64, 48 (1973) [Sov. Phys.-JETP 37, 28 (1972)].

21. W. G. Unruh, "Second Quantization in the Kerr Metric" (preprint, University of California-Berkeley, 1974).

22. L. H. Ford, Ph.D. dissertation, Princeton University, in preparation.

23. S. W. Hawking, "Black Hole Explosions?" (preprint, University of Cambridge, 1974).

24. L. Parker and S. A. Fulling, Phys. Rev. D 7, 2357 (1973).

25. L. Parker, B. L. Hu, and S. A. Fulling, work in progress.

26. B. S. DeWitt, Phys. Rev. 90, 357 (1953) (abstract); ibid. 162, 1239 (1967).

27. R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962).

28. L. Parker and S. A. Fulling, Phys. Rev. D 9, 341 (1974).

29. S. A. Fulling and L. Parker, "Renormalization in the Theory of a Quantized Scalar Field Interacting with a Robertson-Walker Spacetime" (preprint, University of Wisconsin-Milwaukee, 1974).

30. S. A. Fulling, L. Parker, and B. L. Hu, paper in preparation.

31. S. A. Fulling, Phys. Rev. D 7, 2850 (1973).

32. W. G. Unruh, Proc. Roy. Soc., to appear.

33. C. W. Misner, Phys. Rev. D 8, 3271 (1973).

34. B. Chakraborty, J. Math. Phys. 14, 188 (1973).

35. A. D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967) [Sov. Phys.- Dokl. 12, 1040 (1968)].

36. E. Kasner, Am. J. Math. 43, 217 (1921).

37. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, New York, Academic Press, 1965.

BIOGRAPHICAL SKETCHES

Stephen A. Fulling received the A.B. degree from Harvard in 1967 and the Ph.D. in Physics from Princeton in 1972. Since 1972 he has been a Postdoctoral Fellow in the Department of Physics at the University of Wisconsin-Milwaukee. He does research in mathematical physics and quantum field theory, especially their applications in the context of general relativity.

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