

# Vacuum nonsingular black hole

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## Summary

The spherically symmetric vacuum stress-energy tensor with one assumption concerning its specific form generates the exact analytic solution of the Einstein equations which for large  $r$  coincides with the Schwarzschild solution, for small  $r$  behaves like the de Sitter solution and describes a spherically symmetric black hole singularity free everywhere.

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Tiny fish is better than a big cockroach  
Russian folklore

The year 1917 went down in history not only as the year when Lenin seized power in Russia to put the Marx doctrine into practice but also as the year when de Sitter published his cosmological solution<sup>1</sup>

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{r_0^2}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where  $r_0^2 = 3/\Lambda$ , with the cosmological constant  $\Lambda$  responsible for the geometry. During several decades the physical essence of this solution remained obscure. In modern physics it has been mainly used as a simple testing ground for developing the quantum field techniques in curved space-time.

In fifty years people understood<sup>2,3,4</sup> that the de Sitter geometry is generated by a vacuum with nonzero energy density  $\varepsilon = \Lambda c^4/8\pi G$ , described by the stress-energy tensor

$$T_{\alpha\beta} = \varepsilon g_{\alpha\beta}, \quad (2)$$

with the equation of state

$$p = -\varepsilon. \quad (3)$$

From the conservation equation  $T_{;\beta}^{\alpha\beta} = 0$  it follows that  $\varepsilon = \text{const}$  hence  $p = \text{const}$  and the stress-energy tensor (2) describes the isotropic vacuum.

At the beginning of the 80's it became known that a physical state with the stress-energy tensor (2) and equation of state (3) can arise in Grand Unified Theories at very high densities corresponding to the characteristic GUT energy  $\sim 10^{15}\text{GeV}$ . Now this state and the de Sitter geometry are broadly used in the cosmological inflationary scenarios describing the very early epochs in the history of the Universe.<sup>5</sup>

Here I shall try to show how the de Sitter solution and the physics underlying it can shed some light also on one of the most dramatic physical problems - the problem of singularities.

The vacuum is defined as such a kind of matter which does not allow any preferred reference frame connected with it. Therefore any reference frame is comoving with the vacuum, and this property holds not only for the standard vacuum  $T_{\alpha\beta} = 0$  but also for the isotropic vacuum described by the stress-energy tensor (2)<sup>2,3,4</sup>. There exists however another possibility.

In the spherically symmetric case  $T_2^2 = T_3^3$  with all the mixed spatial components equal to zero. The stress-energy tensor with the canonical algebraic form

$$T_2^2 = T_3^3 \quad \text{and} \quad T_0^0 = T_1^1 \quad (4)$$

according to the Petrov algebraic classification<sup>7</sup> has an infinite set of comoving reference frames<sup>2</sup>. Hence, it can be interpreted as the stress-energy tensor describing the spherically symmetric vacuum. In general this vacuum is anisotropic.

From the Petrov classification scheme it follows that there exists only one type of the algebraic structure of stress-energy tensor describing the spherically symmetric vacuum. Let us consider what geometry can be generated by the spherically symmetric vacuum (4).

The best known spherically symmetric solution of the Einstein field equations is the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (5)$$

where

$$r_g = \frac{2GM}{c^2}, \quad (6)$$

and M is the mass of a source measured by a distant observer. The Schwarzschild geometry describes the gravitational field of a spherical mass in empty space outside the mass. Historically it was found by Schwarzschild as the solution of the Einstein equations for a

point mass<sup>8</sup>. This metric is used to describe the result of a gravitational collapse - a black hole - in the situation when the angular momentum of the collapsed star as well as its electric and magnetic charges are zero. In this case the stress-energy tensor responsible for the geometry is equal zero everywhere except a singular point at  $r = 0$ , where the energy density is infinite<sup>6</sup>. All invariants of the Riemann curvature tensor also tend to infinity at  $r \rightarrow 0$  and the standard notion of space-time geometry loses its sense at the singularity<sup>9</sup>.

Let us show that the spherically symmetric vacuum can generate a black hole solution which is regular at  $r = 0$  and everywhere else.

The general spherically symmetric metric has the form

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (7)$$

Because in our case  $T_0^0 = T_1^1$ , the Einstein field equations give

$$\frac{\partial\lambda}{\partial r} + \frac{\partial\nu}{\partial r} = 0, \quad i.e. \quad \lambda + \nu = f(t).$$

Without losing generality we can rescale time in such a way to get  $\lambda + \nu = 0$ . Now we have to make one assumption concerning the specific form of the stress-energy tensor (4).

If we assume that

$$T_0^0 = T_1^1 = \varepsilon_0 \exp\left(-\frac{r^3}{r_0^2 r_g}\right), \quad (8)$$

where  $r_0$  is connected with  $\varepsilon_0$  by the de Sitter relation

$$r_0^2 = \frac{3c^4}{8\pi G\varepsilon_0}, \quad (9)$$

then the standard formula for the mass<sup>6</sup>

$$m(r) = \frac{4\pi}{c^2} \int_0^r T_0^0 r^2 dr, \quad (10)$$

gives at  $r \rightarrow \infty$  the whole mass  $M$  connected with  $r_g$  by the Schwarzschild relation (6).

Integrating the Einstein field equations with the assumed form of  $T_0^0 = T_1^1$  we obtain the following metric

$$ds^2 = \left(1 - \frac{R_g(r)}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{R_g(r)}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (11)$$

where

$$R_g(r) = r_g \left(1 - \exp\left(-\frac{r^3}{r_*^3}\right)\right), \quad (12)$$

and

$$r_*^3 = r_0^2 r_g. \quad (13)$$

This is the exact spherically symmetric solution of the Einstein equations which for  $r \gg r_*$  practically coincides with the Schwarzschild solution and for  $r \ll r_*$  behaves like the de Sitter solution.

From the Einstein equations we derive the remaining components of the stress-energy tensor

$$T_2^2 = T_3^3 = \varepsilon_0 \left(1 - \frac{3r^3}{2r_*^3}\right) \exp\left(-\frac{r^3}{r_*^3}\right). \quad (14)$$

As follows from (8) and (14), our spherically symmetric vacuum is really anisotropic. The difference between the principal pressures

$$p_k = -T_k^k, \quad (15)$$

corresponds to the well known anisotropic character of evolution of the space-time inside a black hole undergoing a spherically symmetric gravitational collapse<sup>10</sup>. For  $r \ll r_*$  isotropization occurs and the stress-energy tensor takes the isotropic vacuum form (2). When  $r \rightarrow 0$  the energy density tends to  $\varepsilon_0$ . For  $r \gg r_*$  all the components of the stress-energy tensor very quickly tend to zero.

Let us now discuss the main properties of our solution. The difference between  $R_g(r_g)$  and  $r_g$  is  $r_g \exp(-r_g^2/r_0^2)$ . The difference between  $m(r)$  and the Schwarzschild mass  $M$  is

$$\frac{M - m(r)}{M} = \exp\left(-\frac{r^3}{r_0^2 r_g}\right). \quad (16)$$

For an object with the mass of several solar masses and  $\varepsilon_0$  corresponding to the GUT energy  $\sim 10^{15}$  GeV the characteristic radius of the metric (13)  $r_* \sim 10^{-17}$  cm. Then the difference between  $m(r_g)$  and  $M$  is given by  $M \exp(-10^{68}) \sim 10^{-66}$  g! Increase of mass and/or  $\varepsilon_0$  leads to decrease of this quantity.

The metric (11) has two event horizons located approximately at

$$r_+ \approx r_g [1 - O(\exp(-r_g^2/r_0^2))]; \quad r_- \approx r_0 [1 - O(r_0/4r_g)]. \quad (17)$$

Here  $r_+$  is the external event horizon. Because  $g_{00}(r_+) = 0$ , the metric (11) describes an object with the same properties as seen by a distant observer, as those defining a black hole: it does not send any signals outside and interacts with its environment only by its gravitational field. The internal horizon  $r_-$  is the Cauchy horizon<sup>11</sup>. Both  $r_+$  and  $r_-$  are removable singularities of the metric. They can be eliminated by an appropriate coordinate transformation. In the coordinates connected with the freely falling particles the metric takes the Lemaitre type form

$$ds^2 = c^2 d\tau^2 - \frac{R_g(r)}{r} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (18)$$

( $\lim_{r \rightarrow 0} (R_g(r)/r) = 0$ ) that is regular both at  $r_+$  and  $r_-$  as well as at  $r \rightarrow 0$ , but is not complete. To find its maximal analytic extension one should introduce, following Chandrasekhar<sup>11</sup>, the isotropic Eddington-Finkelstein coordinates in which the solution (11) is given by

$$ds^2 = |1 - \frac{R_g(r)}{r}| dudv - r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (19)$$

Its maximal analytic extension is obtained in a standard way and it is similar to that for the Reissner-Nordström space-time with the essential difference that it is regular at  $r = 0$ . So the solution presented here is nonsingular everywhere.

The quadratic invariant of the Riemann tensor  $\mathcal{R}^2 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$  has the form

$$\mathcal{R}^2 = 4 \frac{R_g^2(r)}{r^6} + 4 \left( \frac{3}{r_0^2} e^{-r^3/r_*^3} - \frac{R_g(r)}{r^3} \right)^2 + \left( \frac{2R_g(r)}{r^3} - \frac{9r^3}{r_0^4 r_g} e^{-r^3/r_*^3} \right)^2. \quad (20)$$

For  $r \rightarrow 0$   $\mathcal{R}^2$  remains finite and tends to the de Sitter value  $\mathcal{R}_0^2 = 24/r_0^2$  which naturally appears to be the limiting value of the space-time curvature. All other invariants are also finite.

Now let us return to the switch role played by the vacuum in the singularity problem.

Inevitability of singularity as the final state of a collapsing massive body results from theorems on singularities proved by Penrose, Hawking and Geroch in the second half of the sixties<sup>12</sup> (for the extended list of references see<sup>13</sup>). One of the conditions on which singularity theorems are based is the strong energy condition<sup>13</sup> which states that for every

timelike vector  $u^\alpha$

$$(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)u^\alpha u^\beta \geq 0. \quad (21)$$

This condition guarantees that the matter described by the stress-energy tensor satisfying it does not prevent monotonic decreasing of expansion of congruences of timelike geodesics. In other words, it means that in a gravitational collapse when the gravity becomes dominating it leads to an unlimited contraction.

One way to avoid singularity during a gravitational collapse was proposed by Gliner<sup>2,14</sup> who suggested that at very high densities such that all kinds of particles lose their identity matter undergoes a transition into a vacuum like state described by the stress-energy tensor (2) and the de Sitter geometry (1). The very important feature of the de Sitter geometry is the divergence of the geodesic congruences. In this case gravity acts in such a way that the trajectories of the freely (along geodesics) moving test particles behave so as if they were repulsed from the center<sup>15</sup>. Having this in mind Gliner suggested that if a vacuum like physical state is achieved during a collapse then further contraction could be stopped and such a vacuum like state could be a final state in a gravitational collapse instead of a singularity.

Several people tried to eliminate the singularity by replacing it at the Planck scale curvature by the de Sitter geometry (Starobinsky<sup>16</sup>, Markov<sup>17</sup>, Bernstein<sup>18</sup>, Poisson and Israel<sup>19</sup>). According to Poisson and Israel, the transition from the Schwarzschild space-time to the de Sitter space-time is possible but "it is necessary to interpose a layer of non-inflationary material at the interface". The space-time generated during spherically symmetric gravitational collapse can be described by the Schwarzschild vacuum solution down to the quantum barrier. Below this barrier may exist a layer of "uncertain depth in which the geometry remains effectively classical and governed by field equations of the form

$$G^{\mu\nu} = 8\pi T^{\mu\nu}(\text{vacuum polarization}) \sim R^2 \dots, \quad (22)$$

representing one-loop vacuum polarization effects of the gravitational and other quantized fields"<sup>19</sup>.

Recently Frolov, Markov and Mukhanov<sup>20</sup> have proposed a nonsingular black hole

solution obtained by matching the Schwarzschild metric with the de Sitter metric at some spacelike surface layer located at the radius  $r_1 \ll r_g$ . This matched solution has in general a jump at  $r_1$ . This jump results from the author's ad hoc assumptions that i) in the matching layer of thickness  $\Delta l \sim l_{Pl} \sim 10^{-33}$  cm located at  $r_1 \gg r_{Pl}$  the equation of state  $p=0$ ,  $\varepsilon = 0$  has to change to the equation of state  $p = -\varepsilon$  with  $\varepsilon \sim \varepsilon_{Pl}$ ; ii) in this extremely thin layer a nonzero anisotropic stress-energy tensor has to be created and then its isotropization has to occur, all this during the Planckian time  $\Delta t \sim t_{Pl} \sim 10^{-43}$  sec, i.e. practically by a jump.

The exact analytic solution (11) represents a black hole which contains the de Sitter vacuum world instead of a singularity. The stress-energy tensor responsible for geometry describes a smooth transition from the standard vacuum state at infinity to isotropic vacuum state at  $r \rightarrow 0$  through anisotropic vacuum state in intermediate region, what agrees with the Poisson and Israel prediction concerning "non-inflationary material at the interface". At present such a transitional material cannot be described starting from any fundamental theory describing reality at the microscopic level. Indeed, all quantum fields give rise to a vacuum polarization (22) arising in the course of a gravitational collapse. To obtain the appropriate stress-energy tensor representing effects of vacuum polarization created by all quantum fields in the framework of a microscopic theory, we need an appropriate Lagrangian but unfortunately today nobody knows how to write it down.

On the other hand, the stress-energy tensor (4) undoubtedly describes the spherically symmetric vacuum in the framework of general relativity. Its specific form (8) and (14) has the appropriate asymptotic behaviour at  $r \rightarrow 0$  and  $r \rightarrow \infty$ . It generates the analytic solution having necessary asymptotic properties and representing a spherically symmetric black hole singularity free everywhere. Therefore, it seems to be natural to assume that this tensor describes at the macroscopic level the vacuum polarization arising during a spherically symmetric gravitational collapse.

If the final isotropic vacuum state of matter is achieved at the GUT energies  $\sim 10^{15}$  GeV then the characteristic radius of the metric (11)  $r_* \sim 10^{-17}$  cm for  $M \sim 10M_\odot$ . If this happens at the Planckian energy then  $r_* \sim 10^{-20}$  cm. In both cases the characteristic radius  $r_* \gg r_{Pl}$  and hence the classical solution (11) adequately describes the transition



to the de Sitter like limit. According to the Hawking-Penrose theorems on singularities<sup>12</sup> the energy condition (21) can be violated if the principal pressures are so negative that  $\sum p_k < -\varepsilon$ <sup>9</sup>. For the stress-energy tensor (8) and (14) this inequality becomes valid just as  $r^3$  becomes less than  $2r_*^3/3$  i.e. long before the transition to the de Sitter like and the Planckian scale limits !

In conclusion I would like to note that according to (16) the mass contained under  $r_g$  is determined by

$$m_{in} = M(1 - \exp(-r_g^2/r_0^2)). \quad (23)$$

The difference between  $m_{in}$  and the Schwarzschild mass  $M$  depends on the limiting internal density  $\varepsilon_0$  and this fact makes this fantastically small difference very important because it means that investigation of physical processes occurring near the external event horizon can in principle give some information about processes occurring deeply inside a black hole.

## References

- 1 de Sitter, W., *Mon. Not. R. astr. Soc.*, **78**, 3, 1917
- 2 Gliner, E. B., *Sov. Phys. JETP*, **22**, 378, 1966
- 3 Zeldovich, Ya. B., *Sov. Phys. Uspekhi*, **95**, 208, 1968
- 4 Lightman, A. P., Press, W. H., Price, R. H. and Teukolsky S. A., *Problem book in relativity and gravitation*, Princeton University Press, Princeton, 1975
- 5 Linde, A. D., *Particle Physics and Inflationary Cosmology*, Harwood Academic Press, Geneva, 1990
- 6 Landau, L. D., and Lifshitz, E. M., *Classical Theory of Fields*, Pergamon Press, Oxford, 1975
- 7 Petrov, A. Z., *Einstein Spaces*, Pergamon Press, Oxford , 1969
- 8 Schwarzschild, K., *Sitz. Preuss. Akad. Wiss.*, **189**, 1916
- 9 Misner, C. W., Thorne, K. S., and Wheeler, J. A., *Gravitation*, Freeman, San Francisco, 1973

- 10 Zeldovich, Ya. B., and Novikov, I. D., *The Structure and Evolution of the Universe*, Chicago University Press, Chicago, 1983
- 11 Chandrasekhar, S., *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford, 1983
- 12 Hawking, S. W., and Penrose, R., *Proc. R. Soc. London*, **A314**, 529, 1969
- 13 Hawking, S. W., and Ellis, G. F. R., *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge, 1973
- 14 Gliner, E. B., *Sov. Phys. Doklady*, **15**, 559, 1970
- 15 Tolman, R. C., *Relativity, Thermodynamics and Cosmology*, Oxford Clarendon Press, Oxford, 1969
- 16 Starobinsky, A. A., *Phys. Lett.*, **B91**, 99, 1980
- 17 Markov, M. A., *JETP Lett.*, **36**, 265, 1982; *Ann. Phys.*, **155**, 333, 1984
- 18 Bernstein, M. R., *Bull. Am. Phys. Soc.*, **16**, 1016, 1984
- 19 Poisson, E., and Israel, W., *Class. Quantum Gravity*, **5**, L201, 1988
- 20 Frolov, V. P., Markov, M. A., and Mukhanov, V. F., *Phys. Rev.*, **D41**, 383, 1990