The Effect of Spacetime Curvature on Hilbert Space

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Summary

A new, unified basis for the study of gravitational effects in quantum and classical systems described by linear wave equations has been obtained by means of a covariant generalization of Berry's phase. The new formulation reveals aspects of the interaction of particles with weak gravitational fields hitherto thought to apply only to the stationary, nonrelativistic case. Though the emerging gauge structure is that of general relativity, weak gravitational fields behave as non-local vector fields. Their action on wave functions amounts, in first order, to a phase factor. Some effects predicted in the literature are re-calculated and extended to the fully relativistic regime. New effects are also predicted. The new approach affords a more precise description and a better understanding of particle interferometry as a tool to study gravitation.

It has long been recognized that one of the most promising ways to observe gravitational effects is in a quantum context [1], particularly in those systems that exhibit quantization on a macroscopic scale [2]. On the other hand, recent developments in unified theories of particle interactions and cosmology also emphasize the importance of quantum gravity effects. While a satisfactory description of the latter ones requires a still elusive unification of general relativity and quantum mechanics, the novel perspective propounded below leads to a deeper understanding of the simpler problems mentioned at the outset. The guiding logical sequence unfolds as follows.

General relativity describes gravitational effects in spacetime and explains gravitational interactions as spacetime geometry. Quantum mechanics, on the contrary, employs Hilbert space as the representation space for quantum systems. The geometrical structure of Hilbert space has however remained largely unnoticed until the recent discovery of Berry's phase [3]. The parallel transport of the eigenvectors of a quantum system whose Hamiltonian $H(\lambda)$ depends on a set of slowly varying parameters λ_a leads in fact to a geometrical phase $\gamma(C)$ for every transport path C in the space of the λ's [3]. The evolution of the system is also accompanied by the appearance of Abelian gauge potentials [3] or non-Abelian ones [4] when degeneracy is present. Thus, the gauge potentials, which represent the geometrical structure of parameter space, induce a corresponding structure in Hilbert space which results in Berry's phase. Since gravitational fields endow spacetime with geometrical structure, the latter should also affect Hilbert space *when parameter space and spacetime coincide.* This is shown below for the simpler, but physically important case of weak gravitational fields.

The identifcation of spacetime with parameter space, dictated by relativity, requires a fully covariant description of the evolution of the quantum system itself. Thus, the theory of Berry's phase, which is based on the non-relativistic Schrödinger equation, must be made covariant. This is accomplished by using the proper time formalism of Fock, Nambu, Feynman and Schwinger [5]. With this generalization, the gravitational gauge field can be derived from a covariant geometrical phase and defined in spacetime.

The line of reasoning expounded can be most simply illustrated for scalar particles, but can be generalized to include particles of any spin. Consider then a scalar particle of mass m interacting with a potential $U(x)$ and satisfying the equation

$$(\Box - m^2)\phi(x) = U(x)\phi(x). \tag{1}$$

With the help of the appropriate Green's function $G(x, x')$, Eq.(1) together with Feynman's boundary conditions [6] can be transformed into the equation

$$\phi(x) = e^{-ik\cdot x} - \int d^4x' G(x,x')U(x')\phi(x'). \tag{2}$$

Eq.(1), however, can be also re-cast into the form of a Schrödinger equation

$$i\frac{\partial \Phi}{\partial \tau} = H\Phi \tag{3}$$

by introducing the proper time τ, a relativistic invariant parameter which describes the evolution generated by the relativistic Hamiltonian

$$H \equiv -(\Box - U(x)). \tag{4}$$

The relation among the variables x^μ and τ is fixed by assuming $\Phi(x,\tau) = e^{im^2\tau}\phi(x)$, where $\phi(x)$ satisfies the equation

$$H\phi(x) = -m^2\phi(x)$$

which coincides with Eq.(1). Since, in general, $U(x)$ does not depend explicitely on τ, the type of solution chosen for Eq.(3) does indeed exist[7]. Eq.(3) can also be applied to vector and spinor fields[7].

The covariant generalization of Berry's phase [7] can be obtained by applying Berry's argument [3] to Eq.(3) with a parameter-dependent Hamiltonian, i.e.

$$i\frac{\partial \Phi}{\partial \tau} = H(\lambda_a(\tau))\Phi, \qquad a = 1, 2, \cdots k, \tag{5}$$

and by replacing the adiabatically slow evolution condition with the assumption that

$$\frac{< m''|\frac{\partial H}{\partial \tau}|m' >}{< m'|\frac{\partial H}{\partial \tau}|m' >} \ll 1 \qquad m'' \neq m',$$

where $\frac{\partial H}{\partial \tau} \equiv \frac{\partial H}{\partial \lambda_a}\frac{d\lambda_a}{d\tau}$ and $|m' >$ denotes an instantaneous eigenstate of H with eigenvalue m'^2. After a cycle T, one has

$$\Phi(x,T) = exp(i\gamma(C))exp(im^2T)\Phi(x,0) \tag{6}$$

with a corresponding phase

$$\gamma(C) = i\oint_c < \phi|\frac{\partial}{\partial \lambda_a}|\phi > d\lambda_a. \tag{7}$$

Finally, one obtains

$$\phi(x(T)) = exp(i\gamma(C))\phi(x(0)), \tag{8}$$

where $A^a \equiv < \phi|\frac{\partial}{\partial\lambda_a}|\phi >$ defined in the k-dimensional parameter space is the gauge field of Ref.[4]. The main point of the covariant generalization given is represented by the fact that it is now possible to identify parameter space with ordinary spacetime by setting $k = 4$. The parameters are then the collective coordinates of the system considered and A^a becomes a gauge field in the usual sense.

Now consider the problem of a scalar particle interacting with a gravitational field and satisfying the Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \phi - m^2\phi = 0, \tag{9}$$

where ∇_μ denotes covariant differentiation. In the field free case, Eq.(9) becomes

$$\Box\phi_o - m^2\phi_o = 0. \tag{10}$$

In the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + 0(h^2_{\mu\nu})$, one can rewrite Eq.(9) in the form of Eq.(1) and discard terms of second order and higher in $h_{\mu\nu}$:

$$(\Box - m^2)\phi(x) = U(x)\phi(x), \tag{11}$$

with

$$U(x) \equiv h_{\mu\nu}\partial^\mu\partial^\nu - (\frac{1}{2}h_\mu{}^\mu{}_{,\nu} - h_\nu{}^\mu{}_{,\mu})\partial^\nu.$$

By using Green's function $G(x, x')$ for Eq.(11), one obtains

$$\phi(x) = \phi_o(x) + \int d^4x' G(x, x')U(x')\phi(x'), \tag{12}$$

where ϕ_o satisfies Eq.(10). In the first order approximation, $\phi(x')$ in Eq.(12) can be replaced by $\phi_o(x')$. By substituting the expression of $U(x)$ into Eq.(12) and integrating by parts, one arrives at the first order solution

$$\phi(x) = e^{-i\chi}\phi_o(x) \tag{13}$$

with

$$\chi \equiv -\frac{1}{4}\int_X^x dz^\lambda (h_{\alpha\lambda,\beta}(z) - h_{\beta\lambda,\alpha}(z))J^{\alpha\beta}(z) + \frac{1}{2}\int_X^x dz^\lambda h_{\alpha\lambda}(z)P^\alpha, \tag{14}$$

where the integration is taken along a path from X to the field point x and the operators $J^{\alpha\beta}(z)$ and P^{α} are defined by

$$[J^{\alpha\beta}(z),\phi_o(x)] = J^{\alpha\beta}(z)\phi_o(x) \equiv i((x^{\alpha}-z^{\alpha})\partial^{\beta}\phi_o(x) - (x^{\beta}-z^{\beta})\partial^{\alpha}\phi_o(x))$$

$$[P^{\alpha},\phi_o(x)] = P^{\alpha}\phi_o(x) \equiv i\partial^{\alpha}\phi_o(x).$$

The integration path in Eq.(14) is arbitrary. However when (14) is applied to a particular problem, e.g. interferometry, the integration path must be defined. On identifing X^{μ} with parameter space and appling Eq.(7), one gets

$$\gamma(C) = \frac{1}{4}\oint_C dX^{\lambda}(h_{\alpha\lambda,\beta}(X) - h_{\beta\lambda,\alpha}(X))J^{\alpha\beta}(X) - \frac{1}{2}\oint_C dX^{\lambda}h_{\alpha\lambda}(X)P^{\alpha}. \tag{15}$$

When $\phi_o = Ae^{ik_{\mu}X^{\mu}}$, where A is some normalizing constant, ∂^{α} is replaced by ik^{α} in both $J^{\alpha\beta}$ and P^{α}. By using Stokes' theorem, Eq.(14) becomes [8]

$$\gamma(C) = \frac{1}{4}\int_{\Sigma_c} R_{\mu\nu\alpha\beta}J^{\alpha\beta}d\sigma^{\mu\nu}, \tag{16}$$

where $R_{\mu\nu\alpha\beta}$ is the linearized Riemann curvature tensor. The integration path C is now determined by the evolution curve of the system in spacetime, and the result is manifestly gauge invariant. A generalization to include nonlinear wave equations can also be given. A few applications of (15) or (16) to particle interferometers are highly instructive.

Berry's phase due to the field of the earth is calculated first. If earth is assumed spherical, then its gravitational field may be described by the Schwarzschild metric [9]

$$d\tau^2 = -(1-\frac{2MG}{c^2r})dt^2 + \frac{1}{c^2}(1-\frac{2MG}{c^2r})^{-1}dr^2 + \frac{r^2}{c^2}d\theta^2 + \frac{r^2}{c^2}sin^2\theta d\phi^2, \tag{17}$$

where M represents the earth mass. Eq.(15) is now applied to a quadrangular interferometer of vertices A, B, C, D, in which a beam of particles is split at A and the resulting beams interfere at A again after travelling along the opposite paths ABCDA and ADCBA. Since the particles are assumed to move with constant speed v, the integration over the time portion of the spacetime loop may be reduced to space integrations by choosing the limits of integration appropriately. Assuming that the linear dimension of the interferometer ℓ is such that $\frac{\ell}{R} << 1$, where R is the earth radius, and expanding $\frac{1}{r}$ in the neighborhood of $\frac{1}{R}$ up to the third order term, one obtains

$$\begin{aligned}\Delta\chi &= \frac{MG\kappa^{\circ}}{c^2R^2}\frac{c\ell^2}{v}(cos\alpha - cos\theta) - \frac{3}{2}\frac{MG\kappa^{\circ}}{c^2R^2}\frac{\ell}{R}\frac{c\ell^2}{v}(cos^2\alpha - cos^2\theta) + \\ &+ \frac{MG\kappa}{c^2R^2}\ell^2 cos\alpha cos\theta(cos\alpha - cos\theta) - \frac{2MG\kappa}{c^2R^2}\frac{\ell}{R}\ell^2 cos\alpha cos\theta(cos^2\alpha - cos^2\theta).\end{aligned} \tag{18}$$

The parameters are referred to an orthogonal tern of axes x, y, z with origin at A and unit vector $\hat{z}$ directed as the outward normal to the earth surface. Side $\bar{AB}$ lies in the (xz)-plane and makes the angle α with $\hat{z}$ and an angle β with side $\bar{AD}$ which lies in the (yz)-plane at an angle θ with $\hat{z}$. The following relations can be used to transform Eq.(18) above:

$$\vec{A} = \ell sin\beta \hat{N} \qquad \vec{g} = -\frac{MG}{R^2}\hat{z}, \qquad \ell = \bar{AB} = \bar{AD} \tag{19}$$

where $\hat{N}$ is the normal to the interferometer area A,

$$\hat{N} \cdot \hat{z} = \frac{sin\alpha sin\theta}{sin\beta} \equiv cos\gamma. \tag{20}$$

If in addition the particles are non-relativistic so that

$$\kappa^{\circ} \approx \frac{mc}{\hbar} + \frac{\hbar\kappa^2}{2mc} \qquad \text{with } \kappa = \frac{mv}{\hbar} \tag{21}$$

one finds

$$\begin{aligned} \Delta\chi &= \{\frac{m^2}{\hbar^2\kappa} + \frac{\kappa}{2c^2}(1 + 2cos\alpha cos\theta) - \frac{3}{2}\frac{m^2\ell}{\hbar^2\kappa R}(cos\alpha + cos\theta) - \\ &- \frac{\kappa\ell}{c^2 R}(\frac{3}{4} + 2cos\alpha cos\theta)(cos\alpha + cos\theta)\} \mid \vec{A} \times \vec{g} \mid . \end{aligned} \tag{22}$$

The first term coincides with the corresponding term of Anandan's result [10]. When the interferometer is rotated by 180°, the contribution from this term gives a phase shift

$$\Delta\chi = \frac{2m^2}{\hbar^2\kappa} \mid \vec{A} \times \vec{g} \mid = \frac{4\pi m^2 A g \lambda}{h^2} sin\gamma \quad . \tag{23}$$

which is just what has been observed in the COW experiment [11]. The first part of the second term also agrees with the result of Ref.[10] except for a factor $\frac{1}{2}$, there missing because of the incorrect expansion of κ° used. The remaining terms are new. The ratios of the various terms in (22) are $1 : \frac{v^2}{c^2} : \frac{\ell}{R} : \frac{v^2\ell}{c^2 R}$. When $v = 10^{-5}c$, $\ell = 1cm$, $R = 7 \times 10^8 cm$, one obtains $1 : 10^{-10} : 10^{-6} : 10^{-16}$. The last term in Eq.(22) can thus be dropped. Of the remaining terms, the third one represents a new general relativistic effect which is larger than the special relativistic correction represented by the second term.

As a second example, one may consider the phase due to the rotation of a frame of reference which can be described, from the point of view of a co-rotating observer, by the metric [12]

$$d\tau^2 = -(1 - \frac{\omega^2 r^2}{c^2})dt^2 + 2\frac{\omega}{c^2}(-ydx + xdy)dt + \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{24}$$

with $r^2 \equiv x^2 + y^2$ and where ω is the angular velocity about the z-axis. The interferometer considered above is now rotated, but interference occurs at vertex C opposite to A. Eliminating again the time intergration and choosing appropriate integration limits along the space part of the loop, one arrives at the result

$$\Delta\chi = -\frac{\omega\ell^2}{c}(\kappa^\circ + \kappa\frac{c}{v})sin\alpha sin\theta. \tag{25}$$

By using Eqs.(19)-(21), Eq.(25) becomes

$$\Delta\chi = -\frac{2m\vec{\omega}\cdot\vec{A}}{\hbar} - \frac{\hbar\kappa^2\vec{\omega}\cdot\vec{A}}{2mc^2}, \qquad \vec{\omega} \equiv \omega\hat{z} \tag{26}$$

where the first term agrees with the result of other non-relativistic and relativistic approaches [8,10,11,13-22] and the second term represents a relativistic correction. When ω is the angular velocity of rotation of the earth, the first term agrees with the experimental result of Ref.[23]. For the particular case of the Sagnac effect, beam splitting and interference take place at A. Then

$$\Delta\chi = -\frac{2\omega\ell^2}{c}(\kappa^\circ + \kappa\frac{c}{v})sin\alpha sin\theta, \tag{27}$$

which, with the usual approximations, becomes

$$\Delta\chi = -\frac{4m\vec{\omega}\cdot\vec{A}}{\hbar} - \frac{\hbar\kappa^2\vec{\omega}\cdot\vec{A}}{mc^2}. \tag{28}$$

In summary, the geometry of spacetime affects quantum Hilbert space and generates a gravitational Berry's phase. This has been explicitly calculated in the weak field approximation. Far from being academic, the approach has interesting, practical consequences. First, Eq.(15), rewritten as

$$\gamma(C) = \oint_C dX^\lambda G_\lambda(X, X_0) \tag{29}$$

with X_0 as arbitrary point on C, indicates that weak gravitational interactions are effectively described in quantum problems by the non-local vector potential

$$G_\lambda(X, X_0) \equiv -\frac{1}{4}(h_{\alpha\lambda,\beta}(X) - h_{\beta\lambda,\alpha}(X))[(X^\alpha - X_0^\alpha)k^\beta - (X^\beta - X_0^\beta)k^\alpha] + \frac{1}{2}h_{\alpha\lambda}(X)k^\alpha. \tag{30}$$

The corresponding gauge-invariant field $\mathcal{F}_{\mu\nu} \equiv G_{\mu,\nu} - G_{\nu,\mu}$ satisfies an inhomogeneous wave equation. While this was known to occur for weak, stationary fields and low particle velocities [2], the occurrence of such peculiar behaviour in the time-dependent and relativistic regime is surprising. Once the nature of the source term in the equation for $\mathcal{F}_{\mu\nu}$ is understood, it is conceivable that new, interesting experiments may be devised. For instance, the very existence of a vector potential implies in turn the existence of effects analogous to Faraday's induction. Then the applications of Eq.(15) to the earth field and rotation are just two of the many physical situations that can be studied. Even at this stage, the investigation predicts a gravitational correction in Eq.(22) larger than expected. In addition, the experimental results where available [11,23] are in excellent agreement with the present calculations. Finally, the successful treatment of the rotational case indicates that the gravitational Berry's phase introduced provides the appropriate theoretical basis for both optical and non-optical gyroscopy.

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