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Quantum Gravity, Random Geometry and Critical Phenomena

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Abstract

We discuss the theory of non-critical strings with extrinsic curvature embedded in a target space dimension d greater than one. We emphasize the analogy between $2d$ gravity coupled to matter and non self-avoiding liquid-like membranes with bending rigidity. We first outline the exact solution for strings in dimensions $d < 1$ via the double scaling limit of matrix models and then discuss the difficulties of an extension to $d > 1$. Evidence from recent and ongoing numerical simulations of dynamically triangulated random surfaces indicate that there is a non-trivial crossover from a crumpled to an extended surface as the bending rigidity is increased. If the cross-over is a true second order phase transition corresponding to a critical point there is the exciting possibility of obtaining a well defined continuum string theory for $d > 1$.

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String Theory is a powerful model, capable of unifying the Yang-Mills interactions of matter with the universal interaction of gravity. It softens the short distance (ultraviolet) divergences of Einstein Hilbert gravity by smearing out points to one-dimensional extended *strings*. These strings sweep out two-dimensional Riemann surfaces as they evolve in Euclidean time. In the first quantized description of string theory one may view the string coordinates describing the embedding of the worldsheet in the target spacetime as

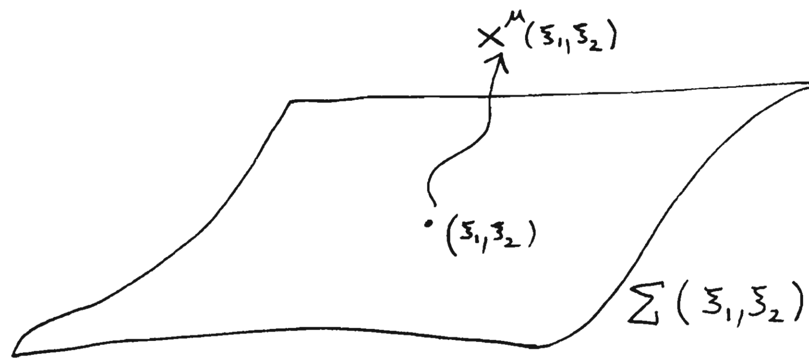

Fig 1. String worldsheet with coordinates X^{μ}.

a collection of scalar fields living on the worldsheet. The worldsheet, however, must fluctuate as one is required to integrate over all admissible metrics to enforce diffeomorphism (reparametrization) invariance. In this way new intrinsic degrees of freedom (the conformal modes of the metric) enter the theory. From the statistical mechanics viewpoint one is thus dealing with an exciting class of models described by certain order fields living on a fluctuating substrate. Averaging over metrics corresponds to being in the universality class of translationally and orientationally disordered fluctuating surfaces or membranes. These are often called liquid-like membranes, as opposed to crystalline or hexatic membranes that are translationally or orientationally ordered respectively [1]. The remarkable fact is that these statistical mechanical models defined on a random mesh are, in a sense, easier to solve than the conventional models defined on a rigid regular lattice. This is because diffeomorphism invariance reduces the number of effective degrees of freedom. It is even possible to admit fluctuations which change the topology of the surface (growth or collapse of handles).

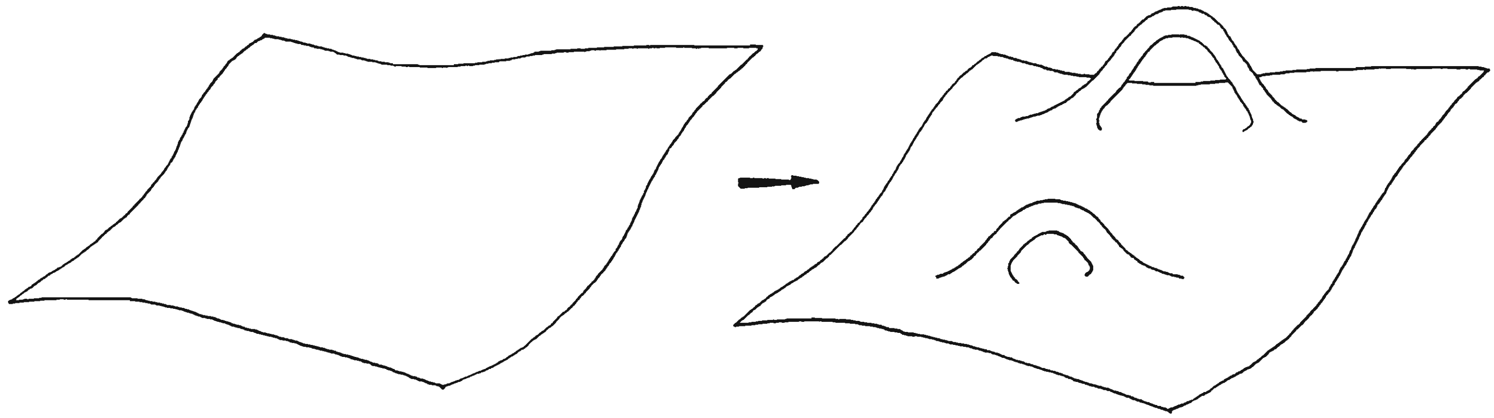

Fig. 2. Fluctuations in the topology of the worldsheet.

This is certainly of great interest as a model of gravity but also provides the basis for an exploration of membranes. Recently particularly simple models corresponding to certain types of conformal matter coupled to $2d$-gravity have been exactly solved including the sum over all possible topologies. The outline of the solution together with the relation to $2d$-gravity and extensions to more realistic random surfaces (flexible liquid-like membranes) is discussed below.

Let us start by considering the extreme case with no matter (order fields) at all. All that remains is the smile on the Cheshire cat - a fluctuating $2d$-surface. This is clearly two-dimensional gravity. Since there are no embedding string coordinates it is also a model of strings in zero dimensions. The Einstein-Hilbert action with a cosmological constant term for $2d$ gravity is

$$S[g] = \frac{-1}{16\pi G} \int_{\Sigma} d^2\xi \sqrt{g} R + \mu \int_{\Sigma} d^2\xi \sqrt{g}, \tag{1}$$

where $g_{\alpha\beta}(\xi_1, \xi_2)$ is the $2d$ metric of the Riemann surface Σ with coordinates ξ_1 and ξ_2.

The partition function Z then depends on two variables, Newton's constant G and the cosmological constant μ

$$Z[G, \mu] = \int [\mathcal{D}g] \; e^{-S[g]} \;, \tag{2}$$

where the path integral is over all admissible metrics of Riemann surfaces Σ. In two dimensions the action (1) is simple since the first term is a topological invariant by the Gauss-Bonnet theorem

$$S = \frac{-\chi(\Sigma)}{4G} + \mu A(\Sigma) \;, \tag{3}$$

where χ is the Euler characteristic of Σ and A is the area. χ is related to the number of handles, or genus h, by $\chi = 2 - 2h$, where for simplicity we are assuming Σ to be closed (without boundaries). The partition function thus reduces to

$$Z[G,\mu] = \sum_h \int dA e^{\frac{\chi}{4G}} e^{-\mu A} \Omega_h(A) \ , \tag{4}$$

where $\Omega_h(A)$ is the density of states of Riemann surfaces Σ of fixed area A and genus h,

$$\Omega_h(A) \equiv \int_{(h;A)} \mathcal{D} g_{\alpha\beta}. \tag{5}$$

$\Omega_h(A)$ is very difficult to calculate as h increases and the sum over genus in (4) diverges [2]. The above expressions are all, in fact, ill-defined. To give them meaning we must regularize the path integrals. One approach is to discretize by replacing Σ by a lattice. A particularly concrete and appealing discretization is to consider all triangulations (or more generally cellular decompositions) of Σ. The surface is thus replaced by a discrete set of n points (vertices) labelled by an index i. The connectivity of the lattice is described by the adjacency matrix

$$C_{ij} = \begin{cases} 1 \text{ if } i \text{ and } j \text{ are connected by a link} \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

This defines a metric on the lattice by fixing all links to have length one. Thus all triangles (cells) in the triangulation are equilateral and of fixed area. The Euler characteristic follows from Euler's relation $\chi = V - E + F$, for V vertices, E edges (links) and F faces (triangles). Local curvature is defined by means of the deficit angle

$$R_i = \frac{\pi}{3} \frac{6 - q_i}{q_i} \quad , \tag{7}$$

where q_i is the coordination number of vertex i

$$q_i = \sum_j C_{ij}. \tag{8}$$

To simulate the integral over metrics the adjacency matrix must be allowed to fluctuate so that the coordination number of a node becomes a dynamical degree of freedom. The local environment of a node is constantly changing. This considerably complicates the study of such models from a computational point of view but also makes them more interesting.

These models are called Dynamically Triangulated Random Surfaces DTRS [3–6]. The basic move to update C_{ij} is a flip on a fundamental parallelogram of two triangles sharing

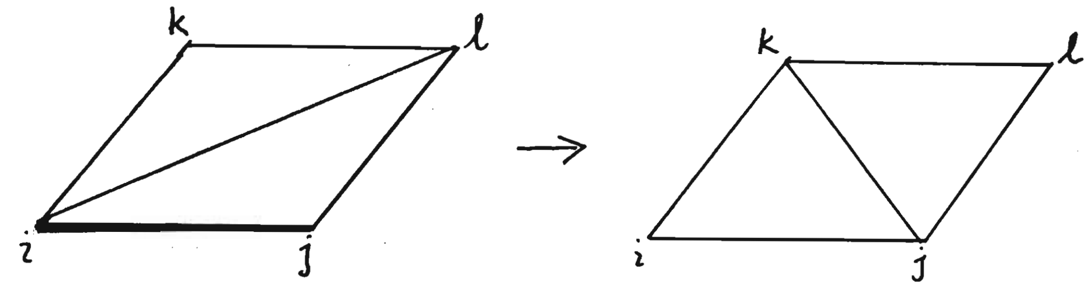

Fig 3. A flip in the triangulation of a surface.

a common edge. The discrete version of the partition function (4) replaces integrals over metrics by sums over admissible triangulations and may be written in the form

$$Z[G,\mu] = \sum_{h=0}^{\infty} e^{\frac{2-2h}{4G}} \sum_{n=0}^{\infty} e^{-\mu n} Z_{h,n} \; , \tag{9}$$

where $Z_{h,n}$ is the number of distinct triangulations with n vertices and genus h. $Z_{h,n}$ is a discrete version of $\Omega_h(A)$, since A is proportional to the n for fixed area elementary triangles. The combinatorial problem of computing $Z_{h,n}$ is related to the quantum field theory problem of calculating the number of distinct Feynman diagrams of a matrix Φ^3

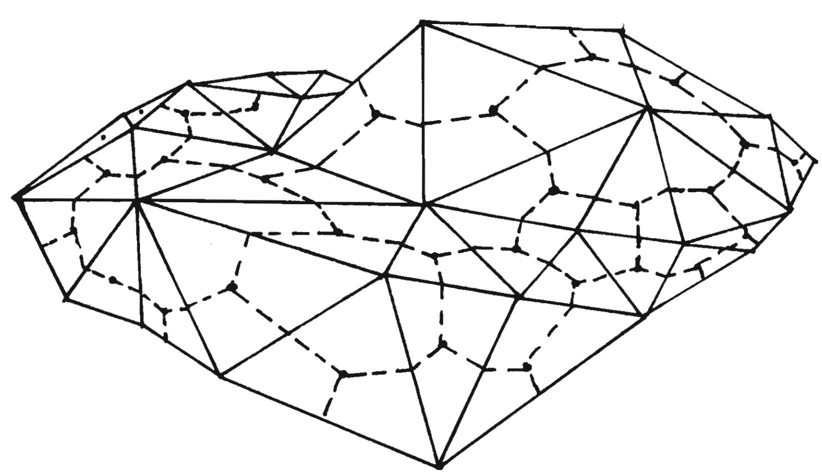

Fig 4. A triangulation of a surface and its dual Φ^3 graph.

field theory. To see this one simply constructs the dual of each triangulation. It may then be shown that the original partition function (9) is related to the solution of the matrix model defined by the integral

$$\zeta(g,N) = \int d^{N^2}\Phi \, \exp\{-N\mathrm{Tr}(\frac{\Phi^2}{2} - \frac{g}{3}\Phi^3)\} \tag{10}$$

over NxN-Hermitian matrices Φ. The exact relation is

$$Z[G,\mu] = \log \zeta(g,N) \quad , \tag{11}$$

where one must identify

$$N = e^{\frac{1}{4G}} \quad \text{and} \quad g = e^{-\mu}. \tag{12}$$

It is necessary to take Φ to be a matrix to generate topologically non-trivial triangulations. In fact it is easy to see that N appears weighted as N^{χ} for a Feynman diagram of Euler characteristic χ in the perturbation expansion of ζ in a double power series in g and N [7].

In the continuum it has been shown [8–10] that there is a scaling relation

$$\Omega_h(A) \sim e^{\mu_c A} A^{\gamma_h - 3}, \tag{13}$$

where the string susceptibility γ_h is

$$\gamma_h = -\frac{1}{2} + \frac{5}{2}h. \tag{14}$$

This result can be generalized to include particular kinds of matter living on the surface. These are the so-called minimal conformal models [11] labelled by two integers p and q. A key parameter of these models is their central charge which measures the response of the free energy to local curvature R of the substrate and is roughly a measure of the number of effective degrees of freedom of the model. For a (p,q) model c is given by

$$c = 1 - \frac{6(p-q)^2}{pq}. \tag{15}$$

Note that c is less than one. Since a single scalar field has $c = 1$ these models of (p,q) conformal matter coupled to $2d$-gravity correspond to strings in less than one target-space dimension. The result (14) for the string susceptibility is more generally

$$\gamma_h = 2 - \frac{(1-h)}{12}\left\{25 - c + \sqrt{(1-c)(25-c)}\right\}. \tag{16}$$

It turns out that pure gravity corresponds to $p = 2$ and $q = 3$. One sees from (15) that $c = 0$ as expected. Near the critical cosmological constant μ_c (or equivalently critical coupling g_c) we see that

$$\int_0^\infty dA e^{-\mu A} \Omega_h(A) \sim (\mu - \mu_c)^{2-\gamma_h} \tag{17}$$

and the mean surface area

$$\langle A \rangle = -\frac{\partial \log Z}{\partial \mu} \tag{18}$$

diverges as $\frac{1}{\mu - \mu_c}$. The string susceptibility γ_h is clearly the critical exponent for the specific heat. Diverging surface area is an indication of criticality. Near μ_c one may thus construct a continuum limit with associated critical exponents that are universal in the sense that they do not depend on the fine details of the lattice. The linearity of γ_h in the genus h implies that $Z[G, \mu]$ is actually a function of only one scaling variable

$$x = (\mu - \mu_c)\ \exp\left\{\frac{1}{4G}\left(1 - \sqrt{\frac{1-c}{25-c}}\right)\right\}. \tag{19}$$

In the Fall of 1989 it was discovered that the complete partition function $Z = Z(x)$ may be determined by taking the so-called double-scaling limit in which $\mu \to \mu_c$ and $N \to \infty$ with $x = (\mu_c - \mu)N^{2m/2m+1}$ held fixed [12–15]. To reach the double-scaling limit for a fixed m requires fine tuning the parameters of a degree $2m$ polynomial potential in the matrix model. The integer m is called the order of multicriticality. The critical behavior at the m^{th} multicritical point is governed by a universal scaling of the density of eigenvalues of the matrix model at the edge of its support [16]. The order of multicriticality is related to the particular conformal matter being coupled by $p = 2$ and $q = 2m - 1$. The specific heat $f(x) = -\partial^2 \ln Z/\partial\mu^2$ is given in this limit by an ordinary nonlinear differential equation of Painlevé I type. For $m = 2$ (pure gravity), for example, it is

$$f^2(x) + \frac{1}{3}f''(x) = x. \tag{20}$$

The string susceptibility determining the behavior of f around the critical point $f \sim (\mu_c - \mu)^{-\gamma_h}$, is given by

$$\gamma_h = -\frac{2}{p+q-1} = -\frac{1}{m}. \tag{21}$$

More general (p, q) models are described by introducing multi-matrix models.

Note that the original matrix integral for pure gravity is unbounded from below at the critical point $g_c = e^{-\mu_c}$ since it corresponds to a cubic potential. There seems to be no escape from this pathology. Pure gravity is still not non-perturbatively well-defined by the matrix model. Models with matter corresponding to m odd are well-defined, however, and this may be a valuable lesson. It may be necessary to add certain classes of matter to $2d$-gravity to render the model non-perturbatively sensible. A non-perturbatively well-defined model may be obtained by introducing a target space supersymmetry in the one-dimensional string [17]. The target space has one time t and one anticommuting $(\theta, \bar{\theta})$ dimension. The total central charge vanishes with the Grassmannian dimension cancelling the $d = 1$ contribution.

Suppose now that we wish to describe more realistic string models corresponding to surfaces embedded in a target space of dimensionality d greater than one. The surface is given by $x^{\mu}(\xi_1, \xi_2)(\mu = 1, ..., d)$. These models obviously have $c > 1$. An immediate problem is then apparent from eqn.(16). According to the continuum results the string susceptibility is imaginary for $1 < c < 25$. This suggests that the model has an inherent instability. A clearer understanding of this instability is gained by examining the continuum limit of the discrete versions of these models with the additional matter action given either by the Nambu-Goto action

$$S_{NG} = \int d^2\xi \sqrt{h} \quad , \tag{22}$$

where h is the determinant of the induced metric $h_{\alpha\beta} = \partial_{\alpha} x^{\mu} \partial_{\beta} x_{\mu}$ and S_{NG} is simply the area of the surface in the induced metric, or by the Polyakov action

$$S_P = \int d^2\xi \sqrt{g} \nabla x^{\mu} \nabla x_{\mu} \quad . \tag{23}$$

Analytical and computational investigations clearly establish that the continuum limit of these models is dominated by surfaces which degenerate into a branched tree of tubes of diameter of order the lattice spacing. These are called branched polymer configurations and are more one-dimensional than two-dimensional.

The origin of these spikes is clear in the Nambu-Goto formulation since an infinitesimally thin long tube has vanishing area and is therefore not suppressed by the area action. The large entropy for such configurations eventually dominates the statistical mechanics of these surfaces. The Polyakov action has been shown to be in the same universality class.

A bending rigidity may be added to the action to suppress branched polymer configurations [18–20]. Consider the extrinsic curvature matrix (Gauss' second fundamental form) K_{ij} given by

$$K^{\mu}_{ij} = D_i D_j x^{\mu}, \tag{24}$$

where D_i is the covariant derivative along the surface. This is the only additional term relevant under rescaling $x \to \lambda x$ that may be added to the string action and so will eventually be generated by radiative corrections in any case. In three dimensions the trace of K is the mean curvature $H = 1/r_1 + 1/r_2$, where r_i are the principal radii of curvature of the surface. The extrinsic curvature action is

$$S_{EC} = \kappa \int d^2\xi \sqrt{g} (\mathrm{Tr} K)^2. \tag{25}$$

Its discrete form may be written as

$$S_{EC} = \kappa \sum_{<ij>} (1 - \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) \quad , \tag{26}$$

where i and j represent triangles that share a common edge and $\hat{\mathbf{n}}_i$ is the unit normal to triangle i. S_{EC} clearly suppresses local fluctuations in the mean curvature of the surface. But the key question is whether there is long-range order in the normals to the surface. The bending rigidity is, in fact, a running coupling — it depends on the scale at which it is measured. A perturbative calculation in the inverse coupling κ^{-1} reveals that strings with bending rigidity are asymptotically free in the same sense as Quantum Chromodynamics. Fluctuations screen the theory and soften the effective bending rigidity as the length scale increases. The momentum p dependence of κ is found to be [18]

$$\kappa^{-1}(p) = \frac{\kappa_0^{-1}}{1 - \frac{d}{2}\frac{\kappa_0^{-1}}{2\pi}\log\frac{\Lambda}{p}} \quad , \tag{27}$$

where Λ is the cutoff or inverse lattice spacing and d is the dimensionality of the target

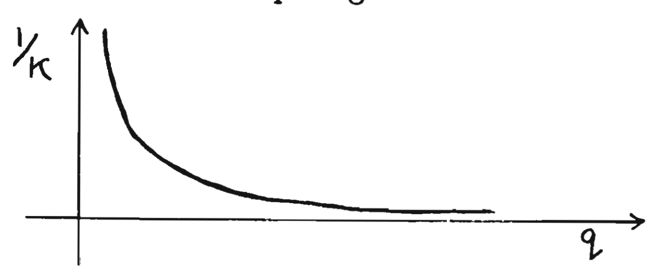

Fig 5. The momentum dependence of the coupling K^{-1}.

space. At large length scales κ tends to zero and there is no suppression of fluctuations in the alignment of normals to the surface. The two-point function decays exponentially

$$< \hat{\mathbf{n}}(\xi_1, \xi_2) \cdot \hat{\mathbf{n}}(0) >= e^{-\frac{|\xi|}{\xi_p}} \tag{28}$$

with persistence length ξ_p. Thus the surface is always disordered or *crumpled* at length scales r exceeding ξ_p. This conclusion is of considerable interest in the study of liquid membranes as well. A typical example of a liquid membrane found in nature (which can also be manufactured in the laboratory) is a lipid bilayer. It consists of two layers of amphiphilic molecules with polar hydrophilic heads and long hydrophobic hydrocarbon tails. These bilayers arrange themselves in thin extended sheets. Within the bilayer individual molecules are quite free to diffuse, so that the in-plane elastic constants turn out to be very low. Another candidate liquid membrane is a monolayer of surfactant molecules of an oil-water interface in a microemulsion. In fact any flexible interface between three-dimensional phases is a candidate system for a liquid membrane model. We see here a beautiful interplay between string theory, quantum gravity and the statistical mechanics

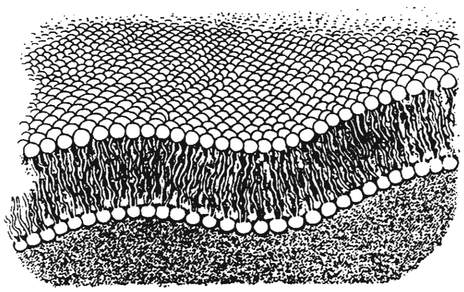

Fig 6. A biological membrane composed of a lipid bilayer.

of fluctuating liquid membranes.

In the last few years such systems have been extensively explored via numerical simulations on a wide range of computers, including parallel machines [21–24]. There are some novel but not fully understood results. The full action which is simulated is given by a quadratic interaction term plus the extrinsic curvature term

$$S = \sum_{<i,j>} (x_i^\mu - x_j^\mu)^2 + \kappa \sum_{[i,j]} (1 - \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) \tag{29}$$

where the first sum is over nearest neighbors and the second over adjacent triangles. For $\kappa < \kappa_c \simeq 1.5$ one sees the expected crumpled surface (see fig. 7). The radius of gyration of these surfaces grows only logarithmically with their area corresponding to infinite Haussdorf dimension d_H defined by

$$R_G^2 \simeq A^{\frac{2}{d_H}} \quad , \tag{30}$$

where R_G is the radius of gyration. For $\kappa > \kappa_c$ the surfaces become extended and considerably smoother with d_H approaching two, which would be the value one would get for a flat surface (see fig. 8). The nature of the cross-over at κ_c is still uncertain. It may be that the system is undergoing a true thermodynamic phase transition. If it is of second order then the continuum limit constructed at the critical coupling would be an interesting string theory corresponding to a real extended $2d$ surface rather than a branched polymer with its largely one-dimensional character. In this case it must be that the coupling $\frac{1}{\kappa}$ ceases to vary with scale (there is a fixed point of the beta function $q\frac{d\kappa}{dq}$) at the critical coupling κ_c. At this point there is said to be a *crumpling transition*. This is the most exciting possibility from the string point of view because it would mean that we have successfully regularized and defined the quantum theory of the string with more than one embedding dimension without any instability arising. The challenge would then be to understand the exact nature of the continuum string theory at the crumpling transition and the origin of the fixed point.

It may also be that the observed cross-over is not a true phase transition and that the persistence length is simply reaching the finite size of the surface that is simulated on the computer. In this case it could still be that the surface is always crumpled on sufficiently large distance scales. This is a real possibility for a liquid membrane but would still leave us without a viable lattice regularization of a string in $d > 1$ dimensions. We are presently performing large-scale simulations in three and four embedding dimensions to decide which of the above possibilities is in fact correct [25].

Finally it is of great interest to extend the technique of dynamically triangulated surfaces to manifolds of higher dimension; in particular to three and four dimensional manifolds. One can then simulate say four dimensional Einstein-Hilbert quantum gravity and seek critical points which provide a non-perturbative definition of a perturbatively non-renormalizable quantum field theory. This would be a very exciting development. Preliminary work indeed seems to indicate that there are indeed phase transitions in $4d$ gravity [26].

Acknowledgements

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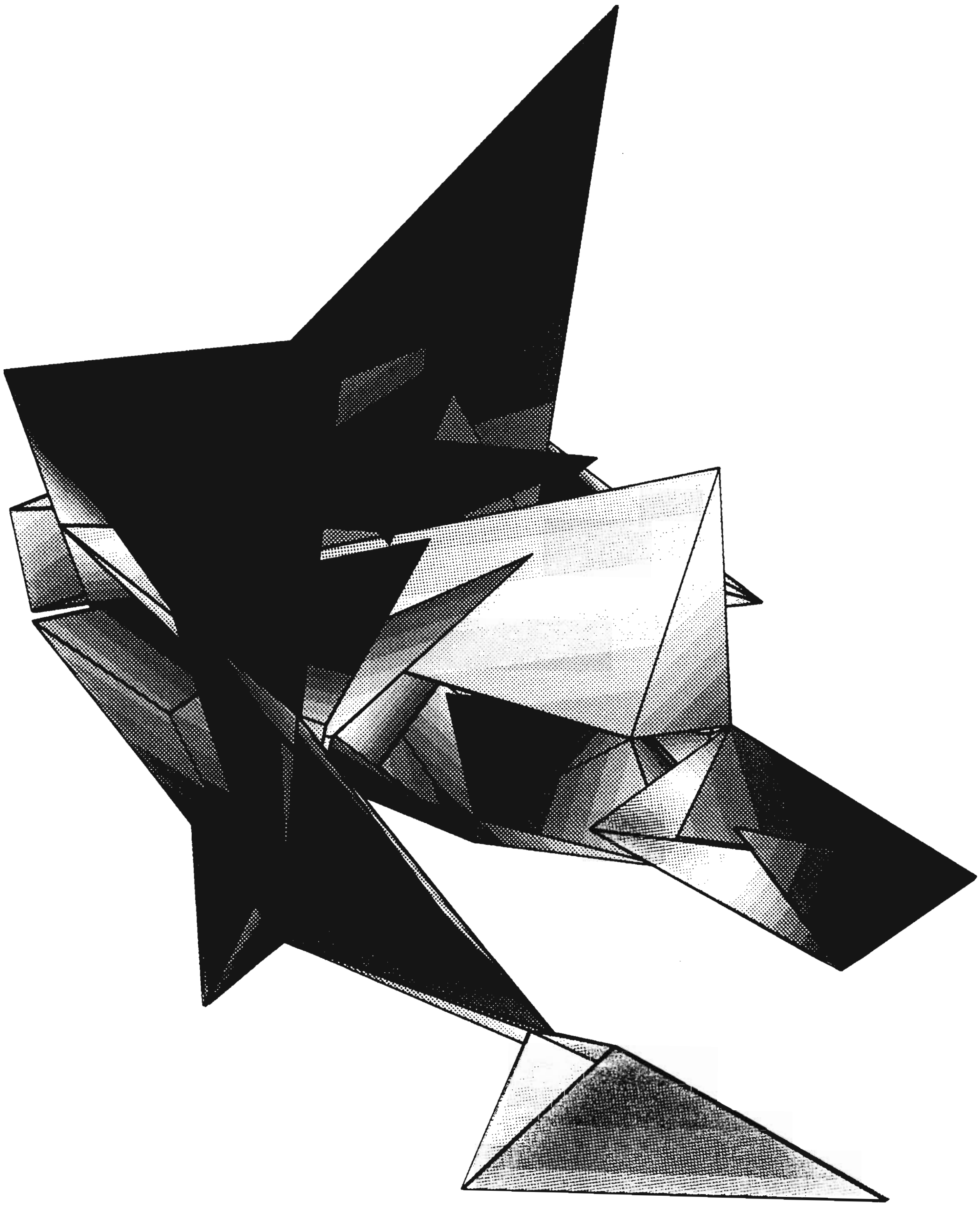

Fig 7. A surface in a crumpled configuration ($\kappa < \kappa_c$)
(figure courtesy of Baillie, Johnston and Williams [22])

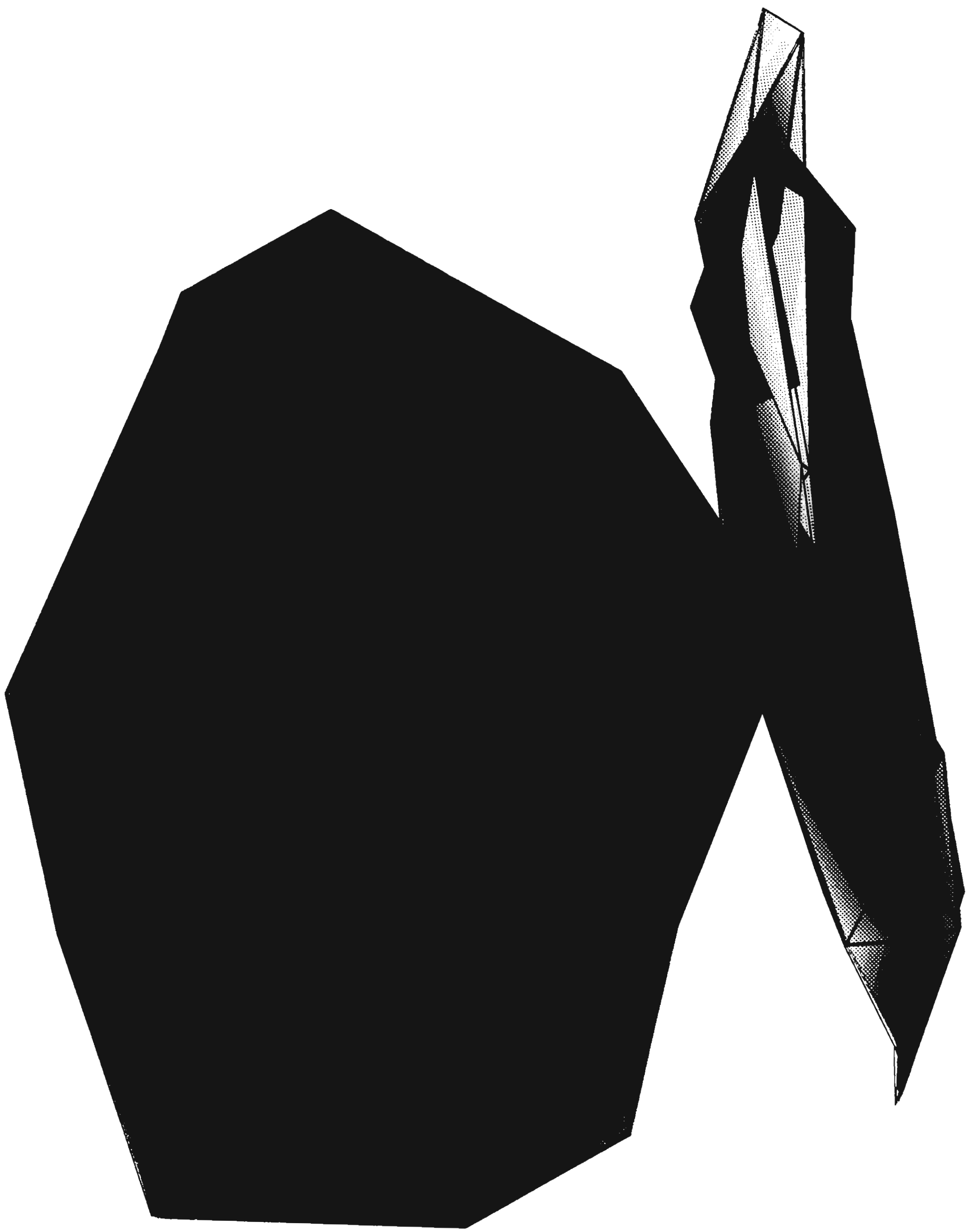

Fig 8. A surface in an extended configuration ($\kappa > \kappa_c$)
(figure courtesy of Baillie, Johnston and Williams [22])

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