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On the question whether fast motion or fast rotation or vibration of an object can decrease the effect of gravity on it

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[illegible] may decrease the gravitational acceleration of its center. The velocity-dependence of the gravitational acceleration of a free particle would seem to substantiate [illegible] claim. This velocity dependence may be used not only [illegible] neutralizing the pull of gravity toward heavy bodies, but [illegible] changing it into a repulsion. For a fast-rotating [illegible] a blind application of the formulas would seem to predict [illegible] gravitational acceleration as the speed of [illegible] the speed of light.

[illegible] Einstein's [illegible] argument somehow must break down [illegible] rotating or vibrating objects. We show here how the velocity dependence of the gravitational acceleration and the absence of an effect of the speed of parts of an object on the acceleration of its center are consistent with each other. Our method of calculation can also be used for explaining the various "experimental proofs" of Einstein's [illegible]

[illegible] principle is not postulated. Nevertheless also for it the gravitational acceleration of the center of a rotating body seems to be substantially (though probably not rigorously) independent of the speeds of the outer parts of the body.

The linear theory of gravitation[1,2,3] predicts a velocity dependence of the gravitational pull. It has been asked[4] whether this fact would make the gravitational acceleration of an atomic nucleus of, say, copper different from the acceleration of a hydrogen nucleus, as the latter is a single proton practically at rest, while the former contains nucleons moving fast within it.[5]

Although the linear theory does not assume the validity of the equivalence principle, which would forbid such a dependence of the gravitational acceleration on the nature of the nucleus, nevertheless it has been shown by consideration of an electro-dynamic model of a rotator that the effects of the high speed of the outer parts of a rotator also in the linear theory of gravitation are substantially undone by compensating effects in the mechanism that keeps the rotator together.[2,4]

It is interesting to remark that the velocity dependence of the gravitational acceleration is a feature not merely of the linear theory, but of Einstein's theory as well. Using "isotropic coordinates", the static spherically symmetric gravitational field around a star may be described by a line element given by

$$ds^2 = Q\,(dx^2 + dy^2 + dz^2) - \Omega c^2\, dt^2, \qquad (1)$$

where Q and Ω are functions of $r \equiv \sqrt{x^2 + y^2 + z^2}$ only. A particle moving in this field with a velocity

$$\left.\begin{aligned} \underset{\sim}{v} \equiv d\underset{\sim}{x}/dt \quad [\text{with speed} \quad v &\equiv \sqrt{Q\,(v_x^2 + v_y^2 + v_z^2)} \\ &\equiv \sqrt{Q\,\underset{\sim}{v}^2}\ ] \end{aligned}\right\} \qquad (2)$$

~ means "boldface print"

will experience an acceleration

$$\underset{\sim}{a} \equiv \frac{d\underset{\sim}{v}}{dt} = -\frac{c^2}{2Q}\underset{\sim}{\nabla}\Omega + \frac{\underset{\sim}{v}^2}{2Q}\underset{\sim}{\nabla}Q + \frac{\underset{\sim}{v}}{\Omega}\frac{d\Omega}{dt} - \frac{\underset{\sim}{v}}{Q}\frac{dQ}{dt}, \qquad (3)$$

where

$$\frac{dQ}{dt} = \underset{\sim}{v}\cdot\underset{\sim}{\nabla}Q, \qquad \frac{d\Omega}{dt} = \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\Omega\,. \qquad (4)$$

Einstein's gravitational equations give

$$Q = (1 + \tfrac{1}{2}l)^4, \qquad \Omega = \left(\frac{1 - \frac{1}{2}l}{1 + \frac{1}{2}l}\right)^2, \qquad (5)$$

where $i = \frac{GM}{c^2 r}$, with G = gravitational constant, c = speed of light, and M = mass of the star. In practice (for planets etc.), i is very small, and

$$\left.\begin{aligned} Q \approx 1 + 2i + \ldots, \quad \Omega \approx 1 - 2i + 2i^2 + \ldots, \\ \text{so } \frac{1}{\Omega} - 1 \approx 2i + 2i^2 + \ldots . \end{aligned}\right\} \tag{6}$$

(This approximation is good enough for the following.) Then, by Eq. (3), the gravitational acceleration of a particle at rest is

$$\underset{\sim}{g} \equiv \underset{\sim}{a}_{(v=0)} = -\frac{c^2}{2Q}\underset{\sim}{\nabla}\Omega = -\frac{GM\underset{\sim}{x}}{r^3}(1 - 4i \ldots), \tag{7}$$

and for a moving particle we find[6]

$$\underset{\sim}{a} = \underset{\sim}{g}\left(1 + \frac{v^2}{c^2}\right) - \frac{4\underset{\sim}{v}(\underset{\sim}{v}\cdot\underset{\sim}{g})}{c^2} + \ldots . \tag{8}$$

where we neglect terms of the order $i^2\underset{\sim}{g}$ and $\frac{v^2}{c^2}i\underset{\sim}{g}$, but we neglect no powers of $\frac{v}{c}$ in the terms linear in $\underset{\sim}{g}$.

If we resolve $\underset{\sim}{a}$ and $\underset{\sim}{g}$ into components $\parallel \underset{\sim}{v}$ and $\perp \underset{\sim}{v}$, we find

$$\underset{\sim}{a}_{\parallel} = \underset{\sim}{g}_{\parallel}\left(1 - 3\frac{v^2}{c^2}\right), \tag{9}$$

$$\underset{\sim}{a}_{\perp} = \underset{\sim}{g}_{\perp}\left(1 + \frac{v^2}{c^2}\right). \tag{10}$$

Figure 1 shows this relation for the case $v = c$ (for photons), for which $\underset{\sim}{a}_{\perp} = 2\,\underset{\sim}{g}_{\perp}$. This explains why Einstein's theory gave twice the bending of light rays passing the sun predicted according to Newton's theory.

Equation (9) shows that, for an object approaching or leaving a star ($\underset{\sim}{v} \parallel \underset{\sim}{g}$), <u>the gravitational attraction changes into a repulsion</u> for $v > c\sqrt{\frac{1}{3}} \approx 0.58\,c$. This is not much of a help in interstellar travel, as such high speeds do not occur just after take-off or just before landing. For objects bypassing a star, the effects of $\underset{\sim}{a}_{\parallel}$ before and after passing cancel each other, and only the bending of the path according to Eq. (10) is important. If a synchrotron beam ($v \approx c$) is shot up vertically, it will experience a gravitational acceleration $2g$ up (instead of g down), but this would be difficult to measure.

A blind application of Eqs. (8) - (10) to a rotational motion

$$x \approx A \cos \omega t, \qquad y \approx A \sin \omega t \qquad (12)$$

in a plane passing through the star would seem to yield a gravitational acceleration which on the average would be decreased. If g is in the x direction, we would find (putting $\beta = \frac{v}{c} = A\frac{\omega}{c}$)

$$a_x = g(1 + \beta^2) \quad \beta^2 [illegible] \sin [illegible],$$

$$a_y = [illegible] \beta^2 g \sin [illegible]$$

[illegible]

[illegible]

This would directly contradict the equivalence principle [illegible] of all free objects are [illegible] [illegible] ence, and therefore must [illegible] [illegible] among each other also in different frames of reference independent of whether the objects rotate or not. Therefore the center of the rotator, when at [illegible], should experience the acceleration g, and not $g(1 - \beta^2)$.

How Eq. (8) is invalidated for a rotator [illegible] understood if we derive this equation directly from the equivalence principle. In the gravitational field with the line element (1) in our original world frame of reference Σ, we can introduce a local freely falling frame of reference Σ' near the point $x \approx y \approx t \approx 0$, $z \approx R$, by a transformation

$$\left.\begin{aligned} x' &= x\sqrt{Q}, \qquad y' = y\sqrt{Q}, \qquad t' = t\sqrt{\Omega}, \\ z' &= [illegible] + \frac{1}{[illegible]\sqrt{Q}}\frac{dQ}{dr}(x^2 + [illegible] + \zeta^2) + \frac{c^2}{4\sqrt{Q}}\frac{d\Omega}{dr}t^2, \end{aligned}\right\} \qquad (15)$$

where $\zeta = z - R$. This transformation makes ds^2 of Eq. (1) equal to the Minkowskian

$$ds'^2 = dx'^2 + dy'^2 + [illegible] c^2 dt'^2 \qquad (16)$$

in Σ', except for terms at least quadratic in x', y', z', t' in the coefficients of dx'^2 etc.

In Σ' then a free particle travels according to

$$x' = v_x' t', \qquad y' = v_y' t', \qquad z' = v_z' t', \qquad (17)$$

with constant velocity $\underset{\sim}{v}'$. By transforming back to Σ, we find $\underset{\sim}{x}$ as a function of t. Thence, we calculate $\underset{\sim}{v} = d\underset{\sim}{x}/dt$ and $\underset{\sim}{a} = d\underset{\sim}{v}/dt$. From the latter equations we eliminate $\underset{\sim}{v}'$, and we obtain $\underset{\sim}{a}$ as a function of $\underset{\sim}{v}$. This yields Eq. (3), and hence Eq. (8). The terms quadratic in v are found to arise in this derivation partially from the terms in (15) quadratic in the coordinates, as $x \approx x' = v_x' t' \approx v_x t$, when squared, will contribute to a second time derivative; and partially from the fact that, for instance in $x = x'/\sqrt{Q} = v_x' t \sqrt{\Omega}/Q$, a factor like $\sqrt{\Omega}/Q$ by Eq. (4) introduces another time dependence. All these $\underset{\sim}{v}^2$ contributions to $\underset{\sim}{a}$ therefore arise from the motion of the particle to points $\underset{\sim}{x}' = \underset{\sim}{v}' t'$ away from its starting point.

For a [illegible] rotational or vibrational motion, on the other hand, in Σ' Eq. (17) must be replaced by something different, and, if the center instantaneously was at rest, this on the average will keep the particle where it came from, and in the mean no v^2 contributions to $\underset{\sim}{a}$ will arise. If the center moves, then $\underset{\sim}{a}$ on the average will be the function (3) or (8) of the velocity $\underset{\sim}{v}$ of the center rather than of some jittering parts.

The method of derivation of Eq. (3) proposed here has not only the advantage of being easily adapted for a discussion of oscillators and rotators. This method, which avoids the explicit mentioning of geodesics in curved space which scares people of little mathematical sophistication, can also be used for deriving the formulas for the experimentally verifiable predictions of Einstein's theory. We mentioned already the bending of light (Eq. 10 and Fig. 1). The gravitational red shift follows from [illegible] $t' = t\sqrt{\Omega}$ in Eq. (15): An observer at infinity in Σ, using for comparison an emitter of light of frequency ν_0, observes the frequency ν of a similar emitter falling freely near a star, and emitting the frequency ν_0 with respect to the freely falling frame of reference Σ'. Therefore,

$$\left.\begin{aligned} &\nu/\nu_0 = (dt)^{-1}/(dt')^{-1} = \sqrt{\Omega} \approx 1 - i, \\ &\text{so} \qquad \frac{\delta\nu}{\nu_0} \approx -i = -\frac{GM}{rc^2}. \end{aligned}\right\} \qquad (18)$$

For deriving the perihelion motion of the planets, we must integrate the equation of motion (3). It is easily verified that integrals of motion are given by

$$F = \frac{v^2}{c^2\Omega^2} + 1 - \frac{1}{\Omega} = \frac{Q\,v^2}{c^2\Omega^2} + 1 - \frac{1}{\Omega} \qquad (19)$$

and by

$$\underset{\sim}{\Lambda} = \frac{Q}{\Omega}\,\underset{\sim}{r} \times \underset{\sim}{v}\,. \qquad (20)$$

By considering what values these constants take faraway from the star ($r \to \infty$, $Q \to 1$, $\Omega \to 1$), we find that F and $\underset{\sim}{\Lambda}$ are related to the energy E and the (orbital) angular momentum $\underset{\sim}{L}$ of the planet of rest mass μ by

$$E = \frac{\mu c^2}{\sqrt{1 - F}} = \frac{\mu c^2 \Omega}{\sqrt{\Omega - \frac{v^2}{c^2}}}\,, \qquad (21)$$

$$\underset{\sim}{L} = \frac{E\underset{\sim}{\Lambda}}{c^2} = \frac{\mu\, Q\, \underset{\sim}{r} \times \underset{\sim}{v}}{\sqrt{\Omega - \frac{v^2}{c^2}}}\,. \qquad (22)$$

By the conventional methods, Eq. (8) is now brought into the form

$$i + \frac{d^2 i}{d\varphi^2} = \frac{m^2 c^2}{2\Lambda^2}\frac{d}{di}\left[Q\left(F - 1 + \frac{1}{\Omega}\right)\right], \qquad (23)$$

with $i = m/r = GM/c^2 r$. Using a trial solution

$$i = B(1 + e\cos\Gamma\varphi)\,, \qquad (24)$$

we find

$$B = \frac{m}{a(1 - e^2)} \qquad (25)$$

and, in first approximation,

$$\Gamma \approx 1, \quad \Lambda^2 \approx mc^2 a(1 - e^2), \quad \text{and} \quad F \approx -\frac{m}{a}\,. \qquad (26)$$

The motion in this approximation is along a Kepler ellipse of major semiaxis a and excentricity e. The next approximation yields Einstein's result for the advance of the perihelion per period,

$$\eta = 2\pi\left(\frac{1}{\Gamma} - 1\right) \approx \pi(1 - \Gamma^2) \approx \frac{6\pi G M}{c^2 a(1 - e^2)}\,. \qquad (27)$$

Figure 1.

Relation between $\underset{\sim}{g}$ and $\underset{\sim}{a}$ for $v = c$ so $\underset{\sim}{a}_{\parallel} = -2\underset{\sim}{g}_{\parallel}$, $\underset{\sim}{a}_{\perp} = +2\underset{\sim}{g}_{\perp}$

Footnotes:

1) F.J.Belinfante and J.C.Swihart, Annals of Physics $\underset{\sim}{1}$, 168 (1957)
2) F.J.Belinfante & J.C.Swihart, Annals of Physics $\underset{\sim}{1}$, 1[illegible] (1957).
3) F.J.Belinfante & J.C.Swihart, Annals of Physics $\underset{\sim}{2}$, 81 (1957).
4) Revista Mexicana de Fisica $\underset{\sim}{4}$, 202-205 (1955).
5) Also on less scientific grounds, it has been suggested that rotation of vibration might undo gravity. For instance, Joseph D. Stites expounds this idea in his proposals for shielding from gravity.
6) This particularly simple dependence of $\underset{\sim}{a}$ on $\underset{\sim}{v}$ is due to our choice of isotropic coordinates. With Schwarzschild's coordinates, r and therefore $\underset{\sim}{v}$ and $\underset{\sim}{a}$ have different meanings and therefore depend differently upon each other.