The Reflection, Refraction, and Intensification of Gravitational Waves on Passing Through Matter

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Introduction

Gravitational waves resemble electromagnetic waves in many respects. They travel through space in all directions with the velocity of light, and hence follow Maxwell's Equations in a vacuum, at least for fields that are not intense enough to distort the space-time continuum. On interaction with matter, gravitational waves are split into a reflected and refracted beam, and the angle of incidence equals the angle of reflection. However, these similarities to electromagnetism conceal important differences between the behavior of gravitational and electromagnetic waves. For instance, the optical properties of matter depend on its chemical constitution, while the gravitational properties depend only on the density of matter. Probably the most important difference between gravity and electromagnetism is that gravitational fields and gravitational waves contain a negative energy density, so that work can be obtained from the generation of gravity waves. Because of their negative energy density, gravitational waves are intensified rather than attenuated on passing through a material medium that converts part of the energy of the waves into elastic vibrations and thence into heat.

The system of units used in this paper is one in which the Newtonian universal constant of gravitation and the dielectric constant of a vacuum are equal to unity.

Wave Equations for the Gravitational Fields, and the Electromagnetic Analogy

Since gravitational waves in free space travel with the velocity of light in all directions, the gravitational field must follow the general traveling wave equation:

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = 0 \qquad \text{(c =velocity of light)}$$

and in addition, since no matter is present, the divergence of the gravitational lines of force, $\nabla \cdot G$, must be zero. It can be shown (see appendix I) that if any vector G satisfies these conditions, a vector K can be found such that:

$$\nabla \times G = -\frac{1}{c}\frac{\partial K}{\partial t}$$
$$\nabla \times K = +\frac{1}{c}\frac{\partial G}{\partial t}$$
$$\nabla \cdot G = 0$$
$$\nabla \cdot K = 0$$

so that the gravitational wave follows Maxwell's Equations in free space.

To see how these equations can be modified in the presence of matter, where $\nabla \cdot G = -4\pi\rho$ (ρ =density of matter), let us try to find values of A and B such that:

$$\nabla \times G = -\frac{1}{c}\frac{\partial K}{\partial t} + A(x,y,z,t)$$
$$\nabla \times K = +\frac{1}{c}\frac{\partial G}{\partial t} + B(x,y,z,t)$$
$$\nabla \cdot G = -4\pi\rho$$
$$\nabla \cdot K = 0$$

where **A** and B are zero when ρ is zero.

Taking the divergence of the first equation, we obtain:

$$\nabla \cdot (\nabla \times G) = 0 = -\frac{\partial}{\partial t}\frac{1}{c}(\nabla \cdot K) + \nabla \cdot A = \nabla \cdot A$$

of which a satisfactory solution in A=0.

From the divergence of the second we obtain:

$$\nabla \cdot (\nabla \times K) = 0 = +\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot G) + \nabla \cdot B$$
$$= -\frac{4\pi}{c}\frac{\partial \rho}{\partial t} + \nabla \cdot B$$

By the equation of continuity for matter,

$$-\nabla\cdot(\rho\vec{V}) = \frac{\partial\rho}{\partial t}$$

(V = velocity)

so that $\frac{4\pi}{c}\nabla\cdot(\rho V) + \nabla\cdot B = 0$

or $\nabla\cdot\left(B + \frac{4\pi}{c}\rho V\right) = 0$

of which a perfectly acceptable solution is:

$$B = -\frac{4\pi}{c}\rho V$$

Therefore the gravitational equations in the presence of matter may be written:

$$\nabla\times G = -\frac{1}{c}\frac{\partial K}{\partial t}$$
$$\nabla\times K = +\frac{1}{c}\frac{\partial G}{\partial t} - \frac{4\pi}{c}\rho V$$
$$\nabla\cdot G = -4\pi\rho$$
$$\nabla\cdot K = 0$$

together with the force equation:

$$\frac{d\,\text{Force}}{d\,\text{Volume}} = \rho G$$

Comparing these equations with the corresponding electromagnetic ones;

$$\nabla\times E = -\frac{1}{c}\frac{\partial H}{\partial t}$$
$$\nabla\times H = +\frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\rho_Q\cdot V$$
$$\nabla\cdot E = 4\pi\rho_Q$$
$$\nabla\cdot H = 0$$
$$\frac{d\,\text{Force}}{d\,\text{Volume}} = \rho_Q E$$

we can see that the two sets of equations become identical if we define:

$$\rho = i\rho_Q \quad \text{or} \quad \rho_Q = -i\rho$$
$$G = -iE \quad \text{or} \quad E = iG$$
$$K = -iH \quad \text{or} \quad H = iK$$

so that as far as the interactions of gravitational waves with matter are concerned, masses behave like imaginary charges, the gravitational field like an imaginary electric field, and the "K" field like an imaginary magnetic field.

The importance of these relationships lies in the fact that

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we know the behavior of the electromagnetic field under a wide variety of conditions, both in free space and in its interaction with charges. To find out what the corresponding equations are for gravitational fields in their interaction with matter, it is merely necessary to substitute $-im$ for q, iG for E, and iK for H in the electromagnetic equations.

Reflection and Refraction of Gravitational Waves by Matter

Suppose that a plane sinusoidal gravitational wave, $G = G_0 \sin 2\pi(nt - x/l)$ is passing through a material medium. If no elastic or viscous forces are acting, all the particles in the medium at a given point are subjected to an acceleration equal to G, and their displacement at a given time is:

$$\vec{r} = \int_0^t\int_0^t \frac{d^2\vec{r}}{dt^2}\,dt\,dt = -\frac{\vec{G}_0}{n^2}\sin 2\pi(nt - x/l) = -\frac{\vec{G}}{n^2}$$

There is a corresponding optical phenomenon in which a high-frequency, sinusoidal electric wave, $E = E_0 \sin 2\pi(nt - x/l)$ acts upon a medium containing electrons whose natural resonant frequency is small compared to the frequency of the incident light, so that the restoring forces are negligible compared to the electric field forces. In such a case the motion of the electrons is governed by the equation:

$$d^2\vec{r}/dt^2 = \frac{q}{m} E_0 \sin 2\pi(nt - x/l)$$

whence

$$\vec{r} = \int_0^t\int_0^t \frac{d^2 r}{dt^2}\,dt\,dt = -\frac{1}{n^2} E_0 \sin(2\pi nt - x/l)$$

$$= -\frac{E}{n^2}\frac{q}{m}$$

The motion of the electrons causes secondary electromagnetic waves to be set up which interfere with the primary incident wave to give a single reflected and refracted beam. The angle of refraction is given by Snell's Law: $\frac{\sin i}{\sin r} = \mu$

while $\mathcal{N}^2 = 1 - 2\pi \sum \frac{q^2/m}{\omega^2} = 1 - \frac{1}{2\pi} \sum \frac{q^2/m}{\nu^2}$

in which the summation is carried out over all the charges in a unit volume.

Now, as far as the propagation of gravity waves and their interaction with matter is concerned, mass acts like an imaginary charge, and the gravitational wave like an imaginary electromagnetic wave, according to the equations:

$$m = iq \quad \text{or} \quad q = -im$$
$$G = -iE \quad \text{or} \quad E = iG$$
$$K = -iH \quad \text{or} \quad H = iK$$

By substituting <u>-im</u> for <u>q</u>, <u>iG</u> for <u>E</u> in the equation of motion of free charges in an electromagnetic wave, it can easily be verified that it becomes identical with the equation for the motion of free matter in a gravitational wave. Furthermore, the motion of matter sets up secondary gravitational waves, which interfere with the primary one to give a reflected and refracted beam, and the index of refraction must be the same as for the electric case, with <u>-im</u> for <u>q</u>. So therefore the ~~square of the~~ index of refraction for gravity waves is:

$$\mathcal{N} = 1 + \frac{1}{2\pi} \sum \frac{m}{\nu^2}$$

$$= 1 + \frac{\rho}{2\pi\nu^2}$$

(ρ = density of the medium)

The angle of incidence equals the angle of reflection, as in the optical case, since this law does not involve the electrical characteristics of the medium in the electromagnetic case, and therefore does not involve the mass characteristics in the gravitational case. The coefficient of reflection for normal incidence is given in the gravitational case as for the electromagnetic case by the formula:

$$R = \frac{\mathcal{N} - 1}{\mathcal{N} + 1} \approx \frac{\rho}{4\pi\nu^2}$$

In a practical system of units in which Newton's Universal

Gravitational Constant is not defined as one,

$$N = 1 + \frac{\rho\gamma}{2\pi n^2}$$, (of the order of $1+10^{-8}$ for most substances)

as can easily be proved by dimensional analysis.

Next let us consider a case in which elastic forces are present, and let us take up the case of shearing forces first. Suppose that a plane sinusoidal gravitational wave is traveling in the x-direction while the field vector, **G**, points in the y-direction, i.e. $G = G_y = G_0 \sin 2\pi(nt - x/l)$. The particles in the medium then vibrate in the y-direction, and set up shearing stresses in this direction. The force per unit volume necessary to maintain the medium in a state of shear is given by the equation:

$$\frac{\partial\,\text{Force}}{\partial\,\text{vol.}} = \sigma \frac{\partial^2 y}{\partial x^2}$$ where σ is the shearing coeff.

The gravitational force per unit volume, ρG, must equal the density of matter times its acceleration, plus the necessary force per unit volume to maintain the condition of shear.

Let $y = -A \sin 2\pi(nt - x/l)$

Differentiating twice with respect to distance and twice with respect to time, we obtain:

$$\frac{\partial^2 y}{\partial x^2} = \frac{A}{l^2} \sin 2\pi(nt - x/l)$$

$$\frac{\partial^2 y}{\partial t^2} = A n^2 \sin 2\pi(nt - x/l)$$

Now,

$$\rho G = \rho \frac{\partial^2 y}{\partial t^2} + \sigma \frac{\partial^2 y}{\partial x^2} = A \sin 2\pi\left(nt - \frac{x}{l}\right)\left[\rho n^2 + \frac{\sigma}{l^2}\right]$$

$$= -y\left[\rho n^2 + \frac{\sigma n^2}{c^2}\right]$$ (since $nl = c$ = velocity of light)

so that $$y = -\frac{G}{n^2}\left[\frac{1}{1 + \frac{\sigma}{\rho c^2}}\right]$$

In the absence of elastic forces, $y_0 = -\frac{G}{n^2}$

and the ratio is: $$\frac{y_0}{y} = \frac{1}{n^2} \cdot n^2\left[1 + \frac{\sigma}{\rho c^2}\right] = 1 + \frac{\sigma}{\rho c^2}$$

Now, σ/pc^2, is very small for most substances, of the order of

10^{-9} or smaller, so that it follows that shearing forces represent a very small correction to the motion of the molecules in a medium under the influence of gravitational forces. Similarly, it can be shown that other elastic forces, such as tensional or compressional forces at the edge of the region of vibrating matter, offer an even smaller correction to the motion of the medium. Both these results are related to the fact that the velocity of sound is very small compared to the velocity of light, so that a medium acts perfectly elastic for gravitational waves.

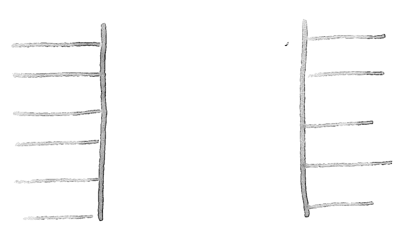
The corrected index of refraction, taking into consideration shearing forces is:

$$\mu = 1 + \frac{2\pi\rho\gamma}{\omega^2}\left[\frac{1}{1+\frac{\sigma}{\rho c^2}}\right] = 1 \hat{+} \frac{\rho\gamma}{2\pi n^2}\left[\frac{1}{1+\frac{\sigma}{\rho c^2}}\right]$$

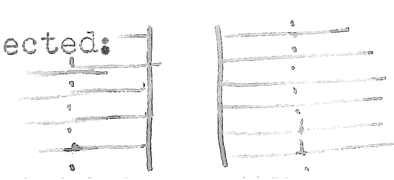
It should be noted that the index of refraction for gravitational waves does not depend on the nature of the molecular and sub-molecular vibrators and rotators in the medium. The reason is that a gravitational field, unlike an electromagnetic field, always exerts a resultant force through the center of mass of the molecule, and hence cannot excite a rotational or vibrational mode. The only exception would occur when the wavelength of the gravitational wave is of the order of the dimensions of the molecule.

Energy Density of Gravity Waves

The energy contained in an electromagnetic wave, $(E^2 + H^2)/8\pi$, per unit volume, is obtained from the principle of conservation of energy, used in conjunction with Maxwell's Equations and the law of force on electric charges, $F=qE$. Similarly, the energy contained in a gravitational wave can be computed from the gravitational wave equations and the law of force for masses, $F=mG$. Such a derivation is given in the appendix for the sake of mathematical rigor, However, the energy density in a gravitational wave can easily be obtained from the fact that the electromagnetic equations become identical with the gravitational ones if we change q to $-\underline{im}$, $\underline{E}$ to $\underline{iG}$, and $\underline{H}$ to $\underline{iK}$. Since the energy density of the electromagnetic wave is $(E^2 + H^2)/8\mathrm{pi}$, it follows that the energy density of a gravitational wave is $(-G^2 - K^2)/8\mathrm{pi}$, so that the gravitational wave has a negative energy density! Although the concept of negative energy is forbidding, it simply means that it takes work to destroy a gravitational field, while while can be obtained during the generation of a gravitational field. As a simple static example of the negative energy density of a gravitational field, consider two infinite plane sheets of equal density parallel to one another. From symmetry, it follows that no lines of force exist between the plates, while the gravitational field is uniform outside, as shown below:



Now, the two sheets attract one another, so that work can be done by allowing the sheets to approach one another. At the same time, the gravitational field occupies a larger volume, while its intensity is unaffected:



Thus it follows that the static gravitational field at least must contain a negative energy density.

Because gravitational waves contain a negative energy density, a gravitational oscillator, such as a vibrating mass, radiates negative energy, and hence slowly but surely gains in kinetic energy. In order for a system to radiate gravitational waves, however, either its center of mass must move (as the center of charge must move in the electric case), or the masses must be separated by a distance of the order of a wavelength. From this it follows that a planetary system like the solar system cannot radiate gravitational waves.

Intensification of Gravitational Waves on Passing through Matter

When gravitational waves pass through matter, part of the energy is converted into vibrational energy of the medium and then into heat. Since the incident gravitational wave originally contained negative energy, and lost energy on traversing the medium, it follows that the wave leaving has still more negative energy, corresponding to a greater intensity of the gravitational wave. Thus gravitational waves are intensified on passing through a material medium.

Since gravitational waves cannot act on molecular or sub-molecular vibrators or rotators, it follows that the dissipation of energy within the medium must be entirely due to bulk frictional forces, which can be lumped together under the viscosity of the medium.

The retarding, viscous forecs per unit volume are given by the equation:

$$\frac{\partial \, \text{Force}}{\partial \, \text{Volume}} = -\eta \nabla^2 \vec{V} \quad [\eta = \text{viscosity}, \vec{V} = \text{velocity}]$$

If the gravitational wave is traveling in the x-direction and all the oscillations occur in the y-direction, then:

$$\frac{\partial\,\text{Force}}{\partial\,\text{Vol}} = -\eta\frac{\partial^2 V_y}{\partial x^2}$$

Now let $y = -A\cos 2\pi(nt - x/l)$

Then:

$$\frac{\partial y}{\partial t} = V_y = +nA\sin 2\pi(nt - x/l)$$

$$\frac{\partial^2 y}{\partial t^2} = n^2 A\cos 2\pi(nt - x/l)$$

$$\frac{\partial V_y}{\partial x} = -\frac{1}{l}\,nA\cos 2\pi(nt - x/l)$$

$$\frac{\partial^2 V_y}{\partial x^2} = -\frac{1}{l^2}\,nA\sin(2\pi)(nt - \tfrac{x}{l})$$

So that: $\quad d(\text{Viscous force})/d\text{Vol.} = +\eta\,\frac{n}{l^2}A\sin 2\pi(nt - \frac{x}{l})$

Now the gravitational force per unit volume equals minus the viscous force plus the density times the acceleration:

$$\rho G = (-1)\eta\frac{n}{l^2}A\sin 2\pi(nt - \tfrac{x}{l}) + n^2\rho A\cos 2\pi(nt - \tfrac{x}{l})$$

$$\therefore\quad G = G_0\cos 2\pi(nt - \tfrac{x}{l} + \phi)$$

$$\text{where } G_0^2 = \left[\frac{\eta^2 n^2}{\rho^2 l^4} + n^4\right]A^2 = \left[\frac{\eta^2 n^6}{\rho^2 c^4} + n^4\right]A^2$$

The rate of dissipation of energy by the viscous forces per unit volume is:

$$-\frac{d^2\,\text{Energy}}{\partial t\,\partial\text{Vol}} = F_{\text{viscous}} \times \vec{V} = \frac{\eta n^2}{l^2}A^2\sin^2 2\pi(nt - x/l)$$

The average value over a cycle is:

$$-\frac{\partial^2\,\text{Energy}}{\partial t\,\partial\text{Vol.}} = \frac{\eta n^2}{l^2}A^2\Big[\overline{\sin^2 2\pi(nt - \tfrac{x}{l})}^{\,\text{average}}\Big] = \frac{\eta n^2 A^2}{2l^2} = \frac{\eta n^4 A^2}{2c^2}$$

$$= \frac{\eta n^4}{2c^2}\left[\frac{G_0^2}{\frac{\eta^2 n^6}{\rho^2 c^4} + n^4}\right]$$

The energy density of the gravitational wave is:

$$I_{\text{ave.}} = \frac{\partial\,\text{Energy}}{\partial\text{Vol}} = \frac{\overline{-G^2 - K^2}^{\,\text{average}}}{8\pi\gamma} = \frac{\overline{-G^2}^{\,\text{average}}}{4\pi\gamma} = -\frac{G_0^2}{8\pi\gamma}$$ [in systems where $\gamma \neq 1$

and

$$\frac{dI}{I\,dx} = -\frac{8\pi\gamma}{G_0^2}\,\frac{\partial^2\,\text{Energy}}{\partial x\,\partial\text{Vol}} = -\frac{8\pi\gamma}{G_0^2}\cdot\frac{\partial^2\,\text{Energy}}{\partial t\,\partial\text{Vol.}}\cdot\frac{dt}{dx} =$$

$$\frac{dI}{I\,dx} = +\frac{8\pi\gamma}{c}\cdot\frac{\eta n^4}{2c^2}\left[\frac{1}{\frac{\eta^2 n^6}{\rho^2 c^4} + n^4}\right] = \frac{4\pi\gamma\eta}{c^3}\left[\frac{1}{\frac{\eta^2 n^2}{\rho^2 c^4} + 1}\right] = \alpha$$

which is the equation for the intensification of gravity waves. The function is of the form: $A\eta/(1+B^2\eta^2)$, and has a maximum for $\eta = 1/B$, or $\eta = \rho c^2/n$. This is around 10^{15} poise, which is in the range of amorphous solids such as glass at room temperature.

For this value of η,

$$(dI/Idx)_{max.} = \frac{4\pi\gamma\rho c^2}{n c^3}\left[\frac{1}{1+1}\right] = \frac{2\pi\gamma\rho}{c n}$$

$$\approx 10^{-18}\,\mathrm{cm}^{-1} \approx 10^{-13}\,\mathrm{mile}^{-1} \approx 1 \text{ per light year}$$

so that even under optimal conditions gravity waves would have to travel through about a light year of solid matter to have their intensity appreciably affected.

Conclusion

(1) Gravitational waves follow Maxwell's equations in a vacuum, and in the presence of matter as well if we define:

$$m = iq, \quad G = -iE, \quad K = iH$$

where K is an auxiliary field which accompanies the gravitational field in the gravitational wave.

(2) Gravitational waves are reflected and refracted on entering a material medium. The angle of incidence equals the angle of reflection, while the coefficient of reflection is about 10^{-7} to 10^{-13} for most substances. The index of refraction is slightly [illegible] than unity by about the same amount.

(3) Elastic forces within a medium have a negligible effect when the medium is subjected to gravitational waves.

(4) Gravitational waves do not affect molecular vibrators or rotators. Hence the interaction of gravity waves with a medium is determined solely by its bulk properties.

(5) Gravitational waves have a negative energy density and hence are intensified rather than absorbed by a material medium. The maximum intensification occurs at very high viscosities such as are found in amorphous solids at room temperature, but even this intensification is so small that the intensity would be doubled only after passing through about a light-year of matter.

APPENDIX

The appendix contains mathematical derivations of statements made without proof in the main body of the essay. Due to the length of the essay proper (2150 words) it is not included for consideration in the contest, but only as a reference in case of doubt as to the validity of statements in the essay.

APPENDIX I

Derivation of Maxwell's Equations from the General Wave Equation.

Given: $$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = 0$$

$$\nabla \cdot G = 0$$

To find: A vector K such that:

$$\left\{ \begin{aligned} \nabla \times G &= -\frac{1}{c}\frac{\partial K}{\partial t} \\ \nabla \times K &= +\frac{1}{c}\frac{\partial G}{\partial t} \\ \nabla \cdot G &= \nabla \cdot K = 0 \end{aligned} \right\}$$

Method:

$$\nabla^2 G = \nabla(\nabla \cdot G) - \nabla \times (\nabla \times G) = -\nabla \times (\nabla \times G).$$

$$\therefore \nabla \times (\nabla \times G) = -\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}$$

Let $A = c\int_0^t (\nabla \times G)\, dt$ [partial integration, holding x, y, z fixed]

then $\nabla \cdot A = c\int_0^t \nabla \cdot (\nabla \times G)\, dt = c\int_0^t 0\, dt = 0$

$$\frac{1}{c}\frac{\partial A}{\partial t} = (\nabla \times G)$$

Substituting this in the equation $\nabla \times (\nabla \times G) = -\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}$,

$$\nabla \times \left(\frac{1}{c}\frac{\partial A}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times A) = -\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}$$

Integrating, $(\nabla \times A) = -\frac{1}{c}\frac{\partial G}{\partial t} + f(x, y, z)$

$$\nabla \cdot (\nabla \times A) = 0 = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot G) + \nabla \cdot f = \nabla \cdot f$$

$\therefore$ $\underline{f}$ is a solenoidal vector point function, and there exist a vector $\underline{u}$ such that:

$$\nabla \times u = f(x, y, z) \qquad [u = \text{function } x, y, z, \text{ not of } t]$$
$$\nabla \cdot u = 0$$

Let $K = A - u$

then $\nabla \times K = \nabla \times A - \nabla \times u = -\frac{1}{c}\frac{\partial G}{\partial t} + f - f = -\frac{1}{c}\frac{\partial G}{\partial t}$

$$\nabla \cdot K = \nabla \cdot A - \nabla \cdot u = 0 - 0 = 0$$

$$\frac{1}{c}\frac{\partial K}{\partial t} = \frac{1}{c}\left(\frac{\partial A}{\partial t} - \frac{\partial u(x,y,z)}{\partial t}\right) = \frac{1}{c}\frac{\partial A}{\partial t} = (\nabla \times G)$$

$$\nabla \cdot G = 0$$

$\therefore$ G and K satisfy the equations desired.

APPENDIX II

Energy Density of the Gravitational Wave

Consider a closed system I, containing a gravitational field and a slightly larger closed system II outside of I.

$$G \cdot \nabla \times K - K \cdot \nabla \times G \equiv \nabla \cdot (K \times G)$$

$$\vec{G} \cdot \frac{1}{c}\frac{\partial \vec{G}}{\partial t} - \frac{4\pi \rho}{c} G \cdot V - \vec{K} \cdot \left(-\frac{1}{c}\frac{\partial \vec{K}}{\partial t}\right) = \nabla \cdot (K \times G)$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\vec{G}\cdot\vec{G}}{2}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\vec{K}\cdot\vec{K}}{2}\right) - \frac{4\pi\rho}{c} G \cdot V = \nabla \cdot (K \times G)$$

$$\frac{\partial}{\partial t}\left(\frac{G^2 + K^2}{8\pi}\right) - \rho \vec{G} \cdot \vec{V} = c\, \nabla \cdot (K \times G)$$

Now applying Green's theorem to the surface of the closed system II,

$$c \iiint_{II} \nabla \cdot (K \times G)\, dVol = c \iint_{II} (K \times G)_n\, dS = c \iint_{II} (K \times 0)_n\, dS = 0$$

Because $G = 0$ on the surface of II, since the G-field lies entirely within the system I (definition of closed system).

$$\therefore \frac{\partial}{\partial t} \iiint_{II} \frac{G^2 + K^2}{8\pi}\, dVol = \iiint_{II} \rho\, \vec{G} \cdot \vec{V}\, dVol.$$

The right hand side is the rate of working of the gravitational energy, which equals the time rate of decrease of the gravitational energy.

$$\therefore \text{Energy} = \iiint_{II} \frac{-G^2 - K^2}{8\pi}\, dVol.$$

$$\frac{d\,\text{Energy}}{d\,Vol.} = \frac{-G^2 - K^2}{8\pi}$$