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Stationary Axisymmetric Space-Times: A New Approach

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ABSTRACT

This essay describes a new approach to the problem of understanding stationary axisymmetric solutions of Einstein's vacuum equations, different from the "Bäcklund transformation" approach which has recently been extensively developed. It translates the problem into one of complex geometry, using the machinery of twistor theory. This, in turn, leads to a procedure which, in principle, generates all solutions. Some explicit examples are presented.

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§1. Introduction

In recent years, there has been much activity in the subject of stationary axisymmetric vacuum space-times, or (more generally) vacuum solutions admitting two Killing vectors. This began with the discovery by Geroch of a transformation which enables one to generate many (he conjectured all) solutions out of a few known ones [1]. His conjecture has been at least partly verified [2]. This approach treats the stationary axisymmetric vacuum equations as a "completely integrable" system, analogous to equations that admit soliton solutions, and uses "Bäcklund transformations" for constructing new solutions out of old [3].

The purpose of this essay is to describe a somewhat different approach, which (roughly speaking) consists of translating the problem into one of complex geometry. It is analogous to Penrose's construction for vacuum spaces with self-dual curvature tensor [4] and it leads to a procedure which enables one to construct (at least in principle) all stationary axisymmetric vacuum solutions. Space-times admitting two spacelike Killing vectors (such as cylindrically symmetric spaces, or colliding plane waves) can be handled by using a slightly different version of the procedure described here.

The relation between this construction and the Bäcklund-transformation approach is as yet unclear, although such a relation presumably exists. The hope is that the new approach described below will lead to a better understanding of the whole problem.

§2. Stationary Axisymmetric Vacuum Spaces

Suppose we have a vacuum space-time admitting two commuting Killing vectors ξ^a and η^a, with ξ^a timelike and η^a spacelike. Suppose further that the determinant $(\xi^a\xi_a)(\eta^b\eta_b) - (\xi^a\eta_a)^2$ of the Killing vectors is not a constant. Then, at least locally, the space-time metric can be put in the form

$$ds^2 = \rho J_{PQ}\, dy^P\, dy^Q - \Omega(d\rho^2 + dz^2), \qquad (1)$$

where y^1, y^2, ρ and z are the space-time coordinates; $\partial/\partial y^1$ and $\partial/\partial y^2$ are the Killing vectors; $\Omega = \Omega(\rho, z) > 0$; and J is a symmetric 2×2 matrix of real-valued functions of ρ and z, with $\det J = -1$. The only coordinate freedom left is that of making constant $SL(2,R)$ transformations on y^1 and y^2.

For example, if $\Omega = 1$, $J = \mathrm{diag}(\rho^{-1}, -\rho)$, $y^1 = t$ and $y^2 = \phi$, then we get the Minkowski metric $ds^2 = dt^2 - dz^2 - d\rho^2 - \rho^2 d\phi^2$ in cylindrical polar coordinates.

Einstein's vacuum equations $R_{ab} = 0$, applied to the metric (1), give

$$\partial_z(J^{-1}\partial_z J) + \rho^{-1}\,\partial_\rho(\rho J^{-1}\partial_\rho J) = 0, \qquad (2)$$

together with equations of the form

$$\partial_z \log\Omega = \text{expression involving } J\text{, but not } \Omega$$

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These last two equations are straightforward to integrate once J is known. Thus, the non-linear equation (2) for the matrix J is the crux of Einstein's equations in this case. What boundary conditions J ought to satisfy is a difficult question: in flat space-time, for example, J is singular on the axis $\rho = 0$, even though the space-time is smooth there.

The remarkable thing about equation (2) is that, although it arises from a curved-space problem, it is in effect a flat-space equation. In fact, it is a special case of an equation in Minkowski space-time, as we shall see in §3. That section goes on to describe how one may characterize all solutions of this more general equation in terms of complex geometry. The description involves twistor theory and complex vector bundles, and the reader who is unfamiliar with these subjects may skip to §4, which describes the application of this theoretical framework to the problem of generating solutions of (2).

§4. A Generalized Equation and Its Twistor Solution

Consider the following equation in Minkowski space-time:

$$\alpha^{A'}\beta^{B'}\partial^{A}{}_{B'}(J^{-1}\partial_{AA'}J) = 0, \tag{3}$$

where $\alpha^{A'}$ and $\beta^{B'}$ are two fixed spinors with $\alpha^{A'}\beta_{A'} \neq 0$. Here $A,B,\ldots$ and $A',B',\ldots$ are 2-component spinor indices [5], and the spinor version $x^{AA'}$ of the standard Minkowski coordinates t,x,y,z is given by

$$\begin{bmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{bmatrix} = \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} .$$

The operator $\partial_{AA'}$ denotes partial differentiation with respect to $x^{AA'}$, and J is a non-singular 2×2 matrix of complex-valued functions of $x^{AA'}$.

If in (3) we put $\alpha^{A'} = (1,0)$, $\beta^{A'} = (0,1)$, and $J = J(\rho,z)$ where $\rho^2 = x^2 + y^2$, then we obtain equation (2); in other words, the solutions of the flat-space equation (3) include all the solutions of the stationary axisymmetric Einstein equations (2).

Notice that if J is a solution of (3), then so is

$$J' = WJV, \tag{4}$$

where $W = W(x^{AA'}\alpha_{A'})$ and $V = V(x^{AA'}\beta_{A'})$ are nonsingular matrices.

Solutions of (3) can be characterized by the following theorem, which involves twistor theory (see [5] for details) and holomorphic vector bundles. In the theorem, R denotes a region in complexified Minkowski space-time, and $\hat{R}$ the corresponding region in twistor space PT. The points $x \in R$ correspond to complex projective lines $\hat{x}$ in $\hat{R}$.

Theorem. There is a natural 1-1 correspondence between

(a) analytic solutions J of (3) on R, modulo the freedom $J \mapsto J'$ as in (4); and

(b) holomorphic rank-2 vector bundles E over $\hat{R}$, such that E restricted to $\hat{x}$ is trivial for all $x \in R$.

Thus, solutions of (3), and hence also of (2), correspond to vector bundles. One way of describing vector bundles is to specify a "patching matrix", and the matrix F appearing in the next section is such a matrix. Its special form reflects the fact that we want solutions of (2), rather than of the more general equation (3).

It is worth remarking that the work described in this essay was inspired by L. Witten's [6] observation that the stationary axisymmetric vacuum equations can be thought of as a special case of the self-dual Yang-Mills equations, for which a twistor construction exists [7].

§4. Constructing Solutions

The general theorem of § 3 leads to the following construction. Let F be a matrix of the form

$$F(\gamma,\zeta) = \begin{bmatrix} f & (-\zeta)^k g \\ \zeta^{-k} g & h \end{bmatrix}$$

such that det F = -1. Here k is an integer, γ and ζ are complex variables, and f,g,h are complex-analytic functions of γ (possibly with singularities), satisfying the reality condition $f(\bar{\gamma}) = \overline{f(\gamma)}$ and similarly for g and h.

Such an F determines a solution J of (2), as follows. Substitute $\gamma = z - \tfrac{1}{2}\rho\zeta + \tfrac{1}{2}\rho\zeta^{-1}$ into F and "split" it:

$$F(z - \tfrac{1}{2}\rho\zeta + \tfrac{1}{2}\rho\zeta^{-1},\zeta) = \hat{H}(\rho,z,\zeta)\, H(\rho,z,\zeta)^{-1}, \qquad (5)$$

where $\hat{H}$ and H are nonsingular 2 x 2 matrices, with H analytic in ζ for $|\zeta| \leqslant 1$ and $\hat{H}$ analytic for $|\zeta| \geqslant 1$ including $\zeta = \infty$. This is analogous to splitting a function into "Taylor" and "Laurent" parts.

Now put

$$J(\rho,z) = P\, H(\rho,z,0)\hat{H}(\rho,z,\infty)^{-1}P, \tag{6}$$

where $P = \text{diag}\,(\rho^{-k/2}, \rho^{k/2})$. Then J is a real symmetric matrix with $\det J = -1$, and it is a solution of equation (2).

§5. Examples and Discussion

The simplest examples correspond to F being diagonal, i.e. $g = 0$. In this case, the splitting (5) is achieved by splitting log f into its Taylor and Laurent parts, and we end up with $J = \text{diag}\,(\rho^{-k}e^{-\psi}, -\rho^{k}e^{\psi})$, where $\psi(\rho,z)$ is an axisymmetric solution of the 3-dimensional Laplace equation $\nabla^2\psi = 0$ given by

$$\psi(\rho,z) = \oint (2\pi i\zeta)^{-1}\, q(z-\tfrac{1}{2}\rho\zeta + \tfrac{1}{2}\rho\zeta^{-1},\zeta)\,d\zeta$$

with $q = \log(-f)$. These are, of course, just the Weyl solutions. For example, $f(\gamma) = -(\gamma+m)/(\gamma-m)$ gives the Schwarzschild solution.

Another example which involves an arbitrary function is given by $k = 1$, $g = e^{-p}$, $f = -\gamma^{-1}\cosh p$ and $h = 2\gamma g$, where $p = p(\gamma)$ is any analytic function. This leads to the family of metrics discussed by Harrison [8].

The meaning of the integer k may be understood by studying the behavior of J on the axis $\rho = 0$. Suppose that f, g and h are analytic in some neighborhood of $\gamma = 0$, and also that $f \neq 0$ at $\gamma = 0$. Then one finds that J_{11} behaves like $-\rho^{-k} f(z)^{-1}$ as $\rho \to 0$, and therefore the norm-squared of the timelike Killing vector $\partial/\partial y^1$ on the axis is $-\rho^{1-k} f(z)^{-1}$. So $k = 1$ gives space-times which are "well-behaved" on the axis, whereas $k \neq 1$ gives solutions with different axis behaviour.

A closer study of the k = 1 case reveals that

$$J = \begin{bmatrix} \rho^{-1} f^{-1}(b^2 - 1) & b \\ b & \rho f \end{bmatrix} + O(\rho^2),$$

where $f = f(z)$ and $b = \frac{1}{2}\rho f \partial_z (f^{-1} g(z))$. So given a metric which is regular on the axis, one can read off what f,g and h have to be (g is only determined up to $g \mapsto g + \lambda f$ where λ is a constant, but this change in F does not alter J). To put this another way, one can specify arbitrary data on the axis, and then use the procedure to find the vacuum solution determined by that data.

The only difficult part of the construction procedure is that of finding the matrices $\hat{H}$ and H which split F. At present, there is no explicit formula for H and H which works in general. But many large classes of metrics can be constructed explicitly using this method, classes which are either well-behaved on the axis (if k = 1) or not (if k ≠ 1). And it follows from the theorem in §3 that all solutions can, at least in principle, be obtained.

Another problem is that of understanding the global geometric structure of these space-times. And here the present method may have the advantage over the "Bäcklund transformation" method (where the geometric aspect is somewhat obscured), because it can be cast in a geometric form, as a construction involving vector bundles (cf. § 3). This aspect has yet to be explored, but it appears to be a promising one.

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