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Mach's Principle and the General Theory of Relativity

by

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Introduction

According to Mach's principle a body has inertia because of some interaction it possesses with the rest of the matter in the universe. One consequence of this principle is that Newton's preferred inertial frames are determined, not by absolute space, but by the motions of distant matter. In particular, a non-rotating frame, in which there are no centrifugal or Coriolis forces, would be a frame in which distant galaxies are non-rotating.

This principle has had a curious history. It was first proposed by Bishop Berkeley, whose contemporaries laughed it to scorn. A hundred and fifty years later it was again proposed, this time by Mach himself. Again it was received with scorn, but this time there was one significant exception. For Einstein was much impressed by Mach's arguments, with their rejection of the unobservable "absolute space", which seemed so in tune with his philosophical standpoint in the special theory of relativity. He coined the phrase "Mach's principle", and gave it physical foundation by his proposal of 1907 that it is the gravitational interaction of a body with the rest of the matter in the universe which gives it inertia. In particular, inertial forces would then be gravitational in origin. In this way Einstein could immediately account for Galileo's discovery that all bodies fall to earth with the same acceleration.* This is most easily seen by using a reference frame falling with a body. In this frame the earth's gravitational force is cancelled by the gravitational (inertial) force of the rest of the universe which is accelerating upwards. This cancellation is complete if the universe has a particular acceleration, which has nothing to do with the properties of the body. Equivalently the acceleration due to gravity, g, depends on the properties of the earth and the universe, but not on those of the body.

This profound insight of Einstein's was followed by an equally profound one a few years later. It is known from special relativity that inertial forces (of which centrifugal and Coriolis forces are just special cases) have a somewhat complicated structure, and are represented by a quantity having 40 independent components (the "three-index" or Christoffel symbols). Now in special relativity we can always choose to use an inertial frame in which the inertial forces vanish everywhere. This implies that the Christoffel symbols are not of the most general kind, but are what is called "integrable". If inertial forces are gravitational in origin, gravitational forces must also be described by Christoffel symbols. But gravitational forces do more than give rise to inertial forces - they also produce the effects, say, of the earth's gravity, which, because of its non-uniformity, cannot be made to vanish everywhere. Accordingly gravitational forces must be described in general by "non-integrable" Christoffel symbols. Now these are just the quantities that are used to describe a curved space or space-time. So Einstein proposed that general gravitational forces are a manifestation of the curvature of space-time.

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* I believe that at the time Einstein was unaware of Eötvös's refined verification of Galileo's discovery.

Einstein's final step was to set up field equations which related the curvature of space-time to the sources of the "gravitational field", that is, to the distribution of matter. These equations then prescribe how much gravitation, or curvature, is produced by a given body or distribution of bodies. It is very striking that one of the possible sets of equations stood out from the others on account of its simplicity (we shall see why later), and this was the one Einstein adopted.

Into the well-known success of these equations we need not enter here. Rather we are concerned with the fate of Mach's principle. By a curious twist of irony it emerged that although Einstein's theory was constructed throughout with this principle in mind, yet nevertheless it was not completely satisfied. For Einstein's field equations have solutions in which bodies possess inertia even though there is no other matter in the universe. Einstein tried to rectify this be tampering slightly with his field equations (adding the so-called cosmical term) but these equations too have anti-Mach solutions. At this stage Einstein seems to have given up Mach's principle.

A Simple Theory of Mach's Principle

In 1953 I published a theory incorporating Mach's principle. In several important respects this theory was oversimplified, but it still, I think, gives physical insight into the problem. The theory accepted Einstein's proposal that inertial forces are gravitational in origin, but it was based on Maxwell-type equations for the gravitational field. The point of this choice of field equations is that it was already known how to express a Mach-type principle in Maxwellian electrodynamics. The principle there takes the form that the electromagnetic field is to be entirely due to physical sources (charges and currents), and not partly due to sources and partly "self-existing" (e. g. by containing a source-free plane electromagnetic wave moving through the universe). Mathematically this principle can be expressed as follows.

In the usual notation Maxwell's equations can be [illegible] the form

$$\Box^2 A^i = [illegible] \qquad (1)$$

$$\frac{\partial A^i}{\partial x^i} = [illegible] \qquad (2)$$

The [illegible] solution [illegible] (1) can be written*

$$A^i = \int \left[\frac{[illegible]}{r}\right]_{ret} dv + B^i$$

where B^i is the general solution of the source-free wave equation

$$\Box^2 A^i = [illegible]$$

[illegible] principle for electromagnetism [illegible] set B^i equal to zero. All the field is then due to the sources. Equation (2) [illegible] by virtue [illegible] the conservation of charge

$$\frac{\partial [illegible]^i}{\partial x^i} = [illegible].$$

By using these equations, with B_i zero, for gravitation, I showed that [illegible] matter was more important than nearby matter in producing inertial forces, and that these forces could have their observed value only if the average density of matter [illegible] the universe ρ was related to the gravitational constant G, and the Hubble constant τ (which determines the rate of expansion of the universe and so its effective radius) as follows:

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* There is [illegible] additional surface integral if the charge distribution extends to infinity.

$$G\rho\tau^2 \sim 1, \tag{3}$$

a relation which is consistent with observation.

A Complete Theory of Mach's Principle

Owing to its simplifications this theory cannot be accepted as a complete theory of Mach's principle. In the last few years there has been much discussion of this principle, and the various proposals have included abandoning it, making topological restrictions on the solutions of Einstein's field equations, and introducing further fields in addition to the Christoffel symbols. I would like to propose here what appears to be the most natural extension of my previous theory, which in addition is in close conformity to Einstein's ideas.

There are two main respects in which my previous theory is inadequate: (i) It is based on a vector potential instead of a tensor potential (only a tensor potential leads naturally to Christoffel symbols). (ii) It is linear, whereas we would expect gravitational energy itself to be a source of gravitation, since, according to Einstein's interpretation of Mach's principle, all forms of energy (or, equivalently, inertia) arise by virtue of interaction with gravitation. (In constrast to this, the electromagnetic field is not a source of electromagnetism, that is, is not charged). When these two limitations are removed we arrive at a theory which is in all respects equivalent to Einstein's.

This remarkable situation arises in the following way. We set up Poisson's equation for a tensor gravitational potential g_{ij}, with a source term consisting of the energy-momentum tensor T_{ij} of matter and the symmetrised energy-momentum (pseudo-) tensor Θ_{ij} of gravitation. We then obtain the equations (somewhat analogous to (1) and (2)):

$$\Box^2 \mathfrak{g}^{ij} = \sqrt{-g}\,(T^{ij} + \Theta^{ij}) \tag{1'}$$

$$\frac{\partial \mathfrak{g}^{ij}}{\partial x^j} = 0 \tag{2'}$$

where $\mathfrak{g}^{ij} = \sqrt{-g}\, g^{ij}$, g is the determinant of g_{ij}, and the operator $\Box^2$ is the Minkowskian one. Now Papapetrou has shown that in a co-ordinate system restricted by the condition (2'), Einstein's field equations can be written in the form (1'), without any approximation whatsoever. This incidentally is the reason why Einstein found his choice of possible field equations to be so restricted: they are the only Poisson-type equations for a self-interacting tensor field.

It is clear now how to proceed. Since the $\Box^2$ operator in (1') is the Minkowskian one, we can integrate (1') to obtain*

$$\mathfrak{g}^{ij} = \int \left[\frac{\sqrt{-g}(T^{ij} + \theta^{ij})}{r} \right]_{ret} dv + B^{ij}$$

where B^{ij} is a solution of the source-free wave equation

$$\Box^2 \mathfrak{g}^{ij} = 0.$$

We now assert that Mach's principle takes the form: set B^{ij} equal to zero. As before, all the field is then due to the sources; now including the gravitational field itself. Equation (2') is satisfied by virtue of the conservation of energy and momentum

$$\frac{\partial}{\partial x^j} \sqrt{-g}(T^{ij} + \theta^{ij}) = 0,$$

which, if we replace T^{ij} by $R^{ij} - \frac{1}{2}R\, g^{ij}$ from Einstein's form of his equations, is equivalent to the contracted Bianchi identities.

By a similar replacement we may write if we wish*

$$\mathfrak{g}^{ij} = \int \left[\frac{\sqrt{-g}(R^{ij} - \frac{1}{2}R\, g^{ij} + \theta^{ij})}{r} \right]_{ret} d\tau \qquad (4)$$

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* There is an additional surface integral if the energy-momentum distribution extends to infinity.

* see previous footnote

which is a non-linear integral equation for g^{ij} (since R^{ij} is a known function of g^{ij}). Any solution of this equation is a solution of Einstein's equations which satisfies Mach's principle. We hence regard Mach's principle as a means of selecting physically satisfactory solutions of Einstein's equations. This is done by choosing suitable boundary conditions, which in any case are needed since in Einstein's form his equations are differential equations. The idea that Mach's principle is a rule for choosing boundary conditions is not new, but hitherto these boundary conditions had not been found.

Future Work

Two major problems remain to be tackled. The first involves equation(2'). There exists a "physicists' proff" that a co-ordinate system can always be found in which(2') holds, but a "mathematicians' proof" is lacking. Secondly, we would like to find out which of the known exact solutions of Einstein's equations satisfy (4). It is easy to see that the Schwarzschild solution does not. The general problem appears to be fairly straightforward. I plan first to test the known cosmological solutions, which appear more likely than the Schwarzschild solution to satisfy Mach's principle, especially as they lead to results like (3). Nevertheless, it may be possible by this means to rule out some of these models or even (the ideal situation) all but one. If this turns out to be so, Mach's principle will make a very specific prediction, which will, we may hope, be eventually confronted by observation. If it survives this test it will at long-last cease to be semi-philosophical, and will take its, surely deserved, place alongside the other great principles of physics.

References

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Summary

The aim of this essay is to propose a new formulation of Mach's principle in general relativity. The principle is regarded as a rule for choosing boundary conditions for Einstein's field equations, which ensures that the inertia of matter is entirely due to its interaction with the rest of the universe. Previous attempts to find such boundary conditions have failed. The new proposal is based on Papapetrou's formulation of Einstein's equations which shows explicitly that the gravitational field itself, as well as matter, is a source of gravitation and inertia. The required boundary conditions can then be stated directly. They lead to a non-linear integral equation which determines all those solutions of Einstein's equations which satisfy Mach's principle.