

NEW APPROACH TO SINGULARITIES.

Essay on the Theory of Gravitation

by

B. G. Schmidt

Hamburg

Summary. One central issue of the theory of Gravitation within the framework of General Relativity are singularities whose definition creates serious difficulties. The essay presents a surprisingly simple and powerful solution of the problem.

To any space time a boundary is attached on which incomplete geodesics terminate as well as inextendable time-like curves of finite length and bounded acceleration. The construction is free of ad hoc assumptions concerning the topology of the boundary and the identification of curves defining the same boundary point. Moreover it is the direct generalisation of the Cauchy completion of positive definite Riemannian spaces.

## A New Approach to Singularities.

Every general relativistic cosmological model in agreement with observations has a singularity. Important theorems by Penrose, Hawking, and Geroch show that these singularities are not a consequence of isotropy assumptions but a general feature of relativistic cosmological models. Hence General Relativity is in trouble, and the difficulties are increased by the fact that no acceptable definition of the term singularity exists.

In this essay a new definition will be proposed which has none of the disadvantages of attempts made so far.

What creates the difficulties in formulating a suitable definition of a singularity in Einsteins theory? The intuitive meaning of the term singularity namely "some quantity goes to infinity" has its origin in classical field theories like electrodynamics. There however a back ground, Minkowski space, is given on which all fields are defined whereas in General Relativity space time with a Lorentz metric is the "background" and at the same time the metric is the very field whose singularities must be defined! Suppose a manifold is given on which the metric tensor is singular at some points or on which a physical quantity like the pressure goes to infinity somewhere. If all these points are removed a space time remains on which every thing is well behaved.

This shows that every definition of a singularity must include a method to decide whether regions have been

removed from space time. Hence the task is to attach somehow to every space time a boundary on which the "real singularities" reside as well as the boundaries of "removed parts".

Space time is a manifold, thus one could incline to construct such a boundary using only the manifold structure. But if for example a point is removed from  $R^2$ , the resulting manifold is diffeomorphic to a cylinder. Nobody however would agree to call a point a boundary of a cylinder! Hence a manifold has no natural boundary and one must try to use the metric of space time to define a boundary.

Suppose for the moment the Riemannian metric were positive definite. Then it would in fact be trivial to find a boundary, because in such a space any two different points  $x$  and  $y$  have a distance  $d(x,y) > 0$ , the infimum of the length of all curves between the two points. Using the distance one defines Cauchy sequences and constructs a completion of the Riemannian space in the same way as the real numbers are constructed from the rational ones. In this way for example a cone gets its top or the interior of a disk in the Euclidian plane a circle as natural boundary.

Unfortunately space time metric is indefinite. No distance between two points is given as any two points can be joined by broken null curves.

The new approach to a definition of a boundary is based on parallel propagation of Lorentz frames. A Lorentz frame

is an orthonormal frame consisting of three ordered spacelike and a timelike vector at a point of space time. Two different frames at one point are related by a Lorentz transformation. The set of all Lorentz frames at all points is a ten-dimensional manifold, called the Lorentz bundle. Four coordinates label the point in space time and six coordinates fix the frame at this point. A family of frames along a curve in space time determines a curve in the Lorentz bundle. If the frames are parallel the curve in the bundle is called horizontal.

A text book result of modern Differential Geometry states that parallel propagation in space time defines ten vector-fields, linearly independent at every point, in the Lorentz bundle. These vector fields contain all information about parallel propagation in space time.

The essential idea is to endow the Lorentz bundle with a positive definite Riemannian metric by defining these vector-fields to be orthonormal. Like any positive definite metric this bundle metric determines a distance  $d(u,v)$  between any two points in the bundle. Therefore one can form the Cauchy completion of the Lorentz bundle and gets a boundary whose points are Cauchy sequences without limit in the bundle.

Can one use this completion of the Lorentz bundle to construct a boundary of space time? Yes, in the following way: Two frames at the same point in space time are related by a Lorentz transformation. Therefore the Lorentz group is a transformation group on the Lorentz bundle. It is possible to establish that this transformation group has a unique

extension on the boundary of the bundle constructed above. A boundary point of space time is now defined as an orbit of the Lorentz group in the boundary of the Lorentz bundle. This boundary of a space time is called the b-boundary ("b" for bundle). The bundle with its boundary is a topological space and this implies that space time together with its b-boundary has a natural topology.

What interpretation can be associated with points of the b-boundary defined in a quite abstract way? Before answering this question it should be mentioned that the b-boundary of a positive definite Riemannian space which <sup>app</sup>arently can be constructed in the same way, coincides with the boundary given by Cauchy completion.

There is a surprisingly simple characterisation of points in the b-boundary, illustrating clearly its physical significance. Fundamental is the interpretation of the length of a horizontal curve, calculated with the bundle metric: Take a horizontal curve in the bundle, in other words a curve  $x(t)$  in space time and a parallelly propagated frame  $e_0(t), \dots, e_3(t)$ . If  $\dot{x}(t)$  is the tangent vector of  $x(t)$ , it is a linear combination

$$\dot{x}(t) = a^0(t)e_0(t) + a^1(t)e_1(t) + a^2(t)e_2(t) + a^3(t)e_3(t)$$

of the frame vectors. Then the length of the curve in the bundle

is

$$L_{(\text{bundle})} = \int \left[ (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 \right]^{\frac{1}{2}} dt.$$

To realise what this means compare this expression with the length of  $x(t)$  in space time which is

$$L_{(\text{space time})} = \int \left[ -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 \right]^{\frac{1}{2}} dt$$

as  $e_0, \dots, e_3$  is a Lorentz frame for which the length of  $e_0$  is  $-1$ , the length of  $e_1, e_2, e_3$  is  $+1$  and all four vectors are orthogonal. Hence the length of the curve in the bundle is the length of  $x(t)$  calculated as if the frame  $e_0, \dots, e_3$  were an orthonormal Euclidian frame. Obviously this length depends on  $x(t)$  as well as on the frame because the frame determines the coefficients  $a^i(t)$  along  $x(t)$ .

Using the meaning of the length of a horizontal curve in the bundle, one easily finds the interpretation of a boundary point of space time. Consider an inextendable horizontal curve of finite length in the bundle. Clearly it determines a boundary point of the bundle and of space time. Hence one finds that an inextendable curve  $x(t)$  in space time defines a boundary point if its "Euclidian length measured in a parallel frame" is finite. The length will depend on the frame, but if it is finite for one frame, it is finite for all frames.

Take for example an incomplete half geodesic in space time. Obviously it terminates at a boundary point as its tangent vector is parallelly propagated. Similar an inextendable time-like curve of finite length and bounded acceleration gives a point of the b-boundary. In this respect the b-boundary essentially differs from former constructions in which points were equivalence classes of incomplete geodesics. This is very important from the physical point of view because of the following reasons: If such a curve exists in a space time, one can in principle build a rocket with finite thrust travelling this path using a finite amount of fuel. After some finite time however this rocket is no more represented by a space time

curve and no prediction can be made concerning its future. Such a situation is now mirrored in the appearance of a boundary point.

Should one call all points of the b-boundary singularities? Such a definition would not be reasonable as it may be possible that the space time under consideration is extendable and some boundary points may become regular points of the extended space time.

Therefore I propose the following definition of a singularity:

A point of the b-boundary of a space time is a singularity if it appears in the b-boundary of every extension of this space time.

Obviously this definition of singularities does not solve any of the problems related with their appearance and physical meaning, but it opens new ways for ~~for~~ further investigations, concerning the nature of singularities and the <sup>o</sup>properties of space times near the singularities. For example: Are singularities related to regions of very high curvature such that it seems necessary to take quantisation into account? Answers to questions of this kind are necessary to develop modifications or generalisations of Einsteins theory which make predictions in situations where Einsteins theory breaks down.

B. G. Schmidt  
I. Institut für Theoretische Physik  
der Universität Hamburg  
2 Hamburg 36  
Jungiusstr.9

Biographical Sketch.

I was graduated in Hamburg where I got the Ph.D. 1968.  
Since 1964 I was doing research in P.Jordans "Seminar  
on General Relativity". On invitation of D.Sciama I spent  
the academic year 1968/69 at the Department of Applied  
Mathematics and Theoretical Physics in Cambridge/England.  
At present I am an assistant at Hamburg University.