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Classification of stationary space-times

by

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Abstract

A systematic approach to the geometric structure of stationary gravitational fields is presented. The algebraic type of the trace-free Ricci tensor together with the propagation properties of the eigenrays in the background 3-space defined by the Killing trajectories serve as a basis for classifying the solutions of the stationary field equations. The eigenrays are the integral curves belonging to the solutions ξ_A of the eigenvalue problem $G_A^B \xi_B = \mu \xi_A$, G_A^B spinor representing the gravitational field in the background space. Many of the already known stationary metrics can be derived in the present scheme but new solutions of the field equations are also obtained. The possible types of the vacuum and electrovac fields are discussed in their connection with the corresponding exact solutions.

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1. Introduction

The content of Einstein's equations of gravitation is that the traces of the curvature tensor (i.e., the Ricci tensor and curvature scalar) are locally determined by the energy-momentum tensor of the matter distribution, whereas the conformal curvature, conveniently described by the Weyl spinor Ψ_{ABCD}, represents the degrees of freedom of the gravitational field itself. This is why the Petrov classification[1,2], based essentially on the algebraic properties of Ψ_{ABCD}, is an adequate means of characterizing the structure of the gravitational field. When the field admits a Killing motion, there is an additional invariant structure present, however. The properties of the Killing vector field, evidently, cannot be described in the framework of the Petrov classification.

In many cases of theoretical importance like the final state of gravitational collapse, certain models of the Universe and almost in all experimental situations (with the possible exception of the gravitational wave experiments) the gravitational field is stationary so that there is a time-like Killing field present. The stationary gravitational field is the subject of this essay. The spinor approach adopted here takes advantage of the presence of a Killing field right at the outset. It has always been an unsettled problem how do the Killing symmetries fit into the general-relativistic spinor formalism. In Section 2 we shall show that a natural spinor description of the stationary fields is available in a three-dimensional Riemannian background space* (being essentially the space of the Killing trajectories[3]).

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*Some of the ideas proposed here have appeared in earlier papers by the author[4,5], but the majority of our results have not been published before. This is also a first attempt at a systematic treatment of the stationary gravitational field problem. Some formal developments in an SU(1,1) spinor approach to space-like Killing motions have been made by Lukács[6] and Perjés[7].

In Section 3 we present the spinor equations of the stationary vacuum. The gravitational field appears on the background space represented by a symmetric spinor G_{AB}. The eigenrays of the gravitational field which have been introduced in earlier papers[5,8] by more sophisticated tensor methods emerge here as the integral curves defined by solutions ξ_A of the eigenvalue problem $G_A^B \xi_B = \mu \xi_A$.

A detailed study of the geometry of eigenrays will be presented in Section 4. This provides us the basis for classifying the stationary space-times. The relation between the eigenrays and the background Ricci tensor of the vacuum as well as the bearances of Petrov classification to the proposed scheme completes the discussion of Sec.4. Finally, Sec.5. is devoted to the structure of stationary electrovac fields. Both well-known stationary metrics and solutions derived recently by use of the present approach are shown to arise under simple assumptions fitting into a coherent picture (Tables 1 and 2).

2. The Killing Spinor

We shall be relying here on the well-known Infeld and van der Waerden version of SL(2,C) spinor theory[2,9]. Our primary object is a one-index spinor* ξ^A which transforms according to the rule

$$\hat{\xi}^A = \Lambda^A{}_B \xi^B, \tag{2.1}$$

$[\Lambda^A_B]$ being an element of the SL(2,C) group. When considering components rather than the abstract geometric objects, it is convenient to introduce the Hermitian connecting quantities $\sigma^\mu_{AC'}$ satisfying

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*Spinor indices take the values 0 and 1. Primed spinor indices (with values 0' and 1') are transformed by the complex conjugate quantities $\bar{\Lambda}^{A'}{}_{B'}$. Spinor indices are raised and lowered by the skewsymmetric spinor ϵ^{AB} and its inverse ϵ_{AB}, respectively. Greek world tensor indices range over 0,1,2 and 3. The signature of the metric is chosen (+---).

$$\sigma_{\mu AC'}\sigma^{\nu}{}_{B}{}^{C'} + \sigma^{\nu}{}_{AC'}\sigma_{\mu B}{}^{C'} = \delta^{\nu}_{\mu}\epsilon_{AB}. \qquad (2.2)$$

Contractions with the time-like Killing vector K^{μ} yield

$$K_{AC'}K_{B}{}^{C'} = \tfrac{1}{2} f \epsilon_{AB} \qquad (2.3)$$

where $K_{AC'} = \sigma^{\mu}_{AC'} K_{\mu}$ is the Killing spinor and $f = K_{\mu}K^{\mu}$ so that in a stationary field $f > 0$.

The Killing spinor has a primed and an unprimed index and, consequently, it can be used for establishing a correspondence between the two kinds of indices. Then, to any given spinor ξ_A one can assign its "adjoint spinor" $\xi^{\pm A}$ by

$$\xi^{+A} \stackrel{\text{def}}{=} \sqrt{\tfrac{2}{f}}\, K^{AA'}\bar{\xi}_{A'}. \qquad (2.4)$$

The advantage of including the factor $\sqrt{\frac{2}{f}}$ in the above definition is that, according to Eq. (2.3), for the double adjoint one can neatly write

$$\xi^{++A} = -\xi^{A}. \qquad (2.5)$$

The complex conjugate of a spinor invariant, say $\xi^{A}\eta_{A}$, is the contraction of the corresponding adjoint spinors:

$$\bar{\xi}^{A'}\bar{\eta}_{A'} = \xi^{+A}\eta^{+}_{A}. \qquad (2.6)$$

The above construction is essentially a covariant form of the SU(2) spinor theory where unprimed upper indices are considered equivalent to primed lower indices[10]: $\xi^{0} \leftrightarrow \bar{\xi}_{0'}$, $\xi^{1} \leftrightarrow \bar{\xi}_{1'}$. In fact one can take, at least locally,

$$\sqrt{\tfrac{2}{f}}\,[K_{AA'}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (2.7)$$

This representation of the Killing spinor gives rise to a restriction of the permissible spin transformations. The subgroup of the transformation matrices $[\Lambda_B^{A}]$ preserving the form (2.7) is precisely the SU(2) group. Using representation (2.7) it can also be seen that the norm $\|\xi\| \overset{def}{=} (\xi^{+A}\xi_A)^{1/2}$ of a one index spinor ξ_A is non-negative:

$$\|\xi_A\|^2 = \xi^{+A}\xi_A = |\xi_0|^2 + |\xi_1|^2 \geq 0. \tag{2.8}$$

In order to avoid formal complications in the forthcoming discussion, it will be advantageous to introduce a special coordinate system in which the Killing field has the components $K^\mu = \delta^\mu_0$. The remaining coordinate freedom is*

$$t' = \Omega t + u(x^i) \tag{2.9a}$$

$$x'^i = x'^i(x^k) \tag{2.9b}$$

with Ω = const. In this coordinate system the metric field quantities appearing in the line element

$$ds^2 = f(dt + \omega_i dx^i)^2 - f^{-1} g_{ik} dx^i dx^k \tag{2.10}$$

are independent of the time coordinate $t = x^0$ and so can be chosen the connecting quantities and Killing spinor $K_{AA'}$.

Quantities g_{ik} transform under (2.9b) like components of a symmetric tensor. Hence we can envisage g_{ik} as the (positive-definite) metric of a three dimensional Riemannian space. In asymptotically flat space-times the parameter Ω is conveniently chosen to be unity. In more general situations the effect of (2.9a) is a conformal rescaling $g_{ik} \to \Omega^2 g_{ik}$ by a constant Ω^2 factor.

By a straightforward generalization of (2.4), all primed spinor indices may be converted into unprimed ones. For instance, the "SU(2) connecting quantities" σ^i_{AB} can be introduced by

$$\sigma^i_{AB} \overset{def}{=} \frac{\sqrt{2}}{f} \sigma^i_{A}{}^{C'} K_{BC'} . \tag{2.11}$$

These quantities have, by Eq. (2.2) the properties

*Lower case Roman indices range over the values 1,2 and 3, and are lowered and raised by g_{ik} and its inverse g^{ik} , respectively. The numerical Levi-Civita symbol is written ϵ_{ijk}.

$$\sigma^{i}_{AB} = \sigma^{i}_{BA} \tag{2.12a}$$

$$\sigma_{i\,C}^{\;A}\sigma_{j\,B}^{\;C} + \sigma_{j\,C}^{\;A}\sigma_{i\,B}^{\;C} = g_{ij}\,\delta^{A}_{B} \tag{2.12b}$$

$$\sigma_{i\,C}^{\;A}\sigma_{j\,B}^{\;C} - \sigma_{j\,C}^{\;A}\sigma_{i\,B}^{\;C} = \sqrt{2}\,i\,\epsilon_{ijk}\,\sigma^{k\,A}_{\;\;B}\sqrt{g}. \tag{2.12c}$$

Equation (2.12c) tells us that the matrices $[\sigma^{i}_{AB}]$ (for a fixed value of i) are essentially generators of the SU(2) group.

From the Hermiticity of the connecting quantities $\sigma^{\mu}_{AC'}$ it follows that $[\sigma^{i}_{AB}]$ matrices are Hermitian. Oddly enough, this is written in a covariant notation as

$$\sigma^{+i}_{AB} = -\sigma^{i}_{AB}. \tag{2.13}$$

The fundamental spinor ϵ_{AB} on the other hand, is self-adjoint;

$$\epsilon^{+}_{AB} = \epsilon_{AB}. \tag{2.14}$$

The curvature tensor of the 3-space with metric g_{ik} can be decomposed into the curvature scalar R and the trace-free part of the Ricci tensor corresponding to a self-adjoint four-index symmetric spinor,

$$R_{ik} - \tfrac{1}{3}g_{ik}R \leftrightarrow \Phi_{ABCD}; \qquad \Phi^{+}_{ABCD} = \Phi_{ABCD}. \tag{2.15}$$

In the presence of the Killing spinor $K_{AA'}$, the algebraic properties are more diversified as compared to the general SL(2,C) formalism. The reason for this is that in a canonical decomposition of the form*

$$\varphi_{(AB\ldots R)} = \alpha_{(A}\beta_{B}\cdots\varrho_{R)} \tag{2.16}$$

it is not only the principal spinors which may be pairwise proportional; specialisations extend to adjoints of principal spinors too. This means that we can have either $\alpha_A \propto \beta_A$ or $\alpha^{+}_{A} \propto \beta_A$ (but $\alpha_A = z\alpha^{+}_{A}$ would imply $\alpha_A = 0$ since $\alpha_A\alpha^{+A} = z\alpha^{+}_{A}\alpha^{+A} = 0$).

*Round index brackets denote symmetrization. For a symmetric spinor, e.g., we have $\varphi_{AB\ldots R} = \varphi_{(AB\ldots R)}$.

3. Spinor derivatives and the eigenrays

The covariant derivatives of 3-tensors are formed by using the 3-metric g_{ik}. A covariant 3-derivative of spinor fields with the standard properties (linearity, reality, Leibnitz rule) can also be defined. We postulate, by analogy with the four dimensional spinor analysis[12],

$$\nabla_i \epsilon_{AB} = 0 = \nabla_i \sigma^k_{\ AB} . \tag{3.1}$$

The spinor affine connection $\Gamma_i{}^B{}_A$, appearing in the covariant derivative

$$\nabla_i \xi_A = \partial \xi_A / \partial x^i - \Gamma_i{}^B{}_A \xi_B \tag{3.2}$$

can be expressed, using (3.1), as follows;

$$\Gamma_i{}^B{}_A = -\frac{1}{2} \sigma_k{}^{BC} \left(\partial \sigma^k_{\ AC} / \partial x^i + \Gamma_{ij}^{\ \ k} \sigma^j_{\ AC} \right). \tag{3.3}$$

Having introduced the spinor derivative operation, we find that the covariant derivative of the quantity $\sqrt{\frac{2}{f}} K^{AA'}$, appearing in the spinor adjoint (2.4), vanishes. That is to say, the operations of covariant derivation and adjunction commute.

The field equations of stationary space-times take a remarkably simple form in the spinor notation. Introducing the complex 3-vector[5]

$$\underline{G} \stackrel{\mathrm{def}}{=} \frac{\nabla f - \nabla \times \underline{\omega} f^2}{2f} , \tag{3.4}$$

we have, for instance the vacuum equations in the tensor form

$$R_{ik} + G_i \overline{G}_k + \overline{G}_i G_k = 0 \tag{3.5a}$$

$$(\nabla - \underline{G} + \underline{\overline{G}}) \cdot \underline{G} = 0 \tag{3.5b}$$

$$(\nabla - \underline{G} + \underline{\overline{G}}) \times \underline{G} = 0 \tag{3.5c}$$

Writing $G_A^B = \sigma^{iB}_{\ \ A} G_i$ and $\nabla_A^B = \sigma^{iB}_{\ \ A} \nabla_i$, these equations are concisely put as

$$\Phi_{ABCD} = G_{(AB} G^+_{CD)} \tag{3.6a}$$

$$\left(\nabla_A^B - G_A^B - G^{+B}_{\ \ A} \right) G_B^C = 0 , \tag{3.6b}$$

where Φ_{ABCD} is the (trace-free) Ricci spinor (cf.(2.15)).

From (3.6a) it is straightforward that Φ_{ABCD} is locally determined* by G_{AB}. The principal spinors of G_{AB} are solutions ξ_A of the eigenvalue problem

$$G_A{}^B \xi_B = \mu \xi_A . \tag{3.7}$$

With the exception of the case $\underline{G}\cdot\underline{G} = 0$, there are two linearly independent eigenspinors ξ_A.

The spinor ξ_A defines a real vector $\underline{l}$ in the 3-space of metric g_{ik}:

$$l^i = \sqrt{2}\, \sigma^i{}_A{}^B \xi^{+A} \xi_B \tag{3.8}$$

(the $\sqrt{2}$ factor is included here for later convenience). It also determines a complex null vector $\underline{m}$ orthogonal to $\underline{l}$ by

$$m^i = \sigma^{iAB} \xi_A \xi_B . \tag{3.9}$$

This complex vector spans a real plane element with normal $\underline{l}$. Thus, a one-index spinor ξ_A defines a complete vector basis in the 3-space. Normalizing to unity, i.e., taking

$$\xi^{+A} \xi_A = 1, \tag{3.10}$$

we obtain a complex vector basis with $\underline{l}\cdot\underline{l} = \underline{m}\cdot\underline{\bar{m}} = 1$ and all other scalar products between basis elements vanishing.

The propagation properties of the vector basis $(\underline{l}, \underline{m}, \underline{\bar{m}})$ are conveniently characterized by the complex rotation coefficients (or, equivalently, SU(2) spin coefficients[5])

$$\kappa = - l_{i;k} l^k m^i, \quad \rho = m_{i;k} \bar{m}^k l^i, \quad \sigma = m_{i;k} m^k l^i, \quad \tau = m_{i;k} \bar{m}^k \bar{m}^i, \quad \epsilon = m_{i;k} l^k \bar{m}^i, \tag{3.11}$$

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*In a general stationary field, with matter present, (3.6a) contains matter field terms too.

denoting covariant derivation by a semicolon in the suffix. The first curvature of the congruence of curves with tangent vector $\underline{l}$ is given by κ . The divergence, curl and complex shear of the congruence is given, respectively by Re ρ , Im ρ and σ .

When ξ_A is chosen a solution of (3.7), the curves with tangent vector $\underline{l}$ may be called the eigenrays of the gravitational field. In static space-times (with $\underline{G}$ real) the eigenrays are the orthogonal trajectories of the equipotential surfaces f = constant in the three-space. In the generic case, however, the geometric picture is more complicated[5] and there may be two solutions of the eigenray problem*. The geometry of eigenrays, as is seen from Eq. (3.7), is closely related to the algebraic type of the gravitational field. These together will serve as a basis for classifying the stationary gravitational fields, in the following sections.

4. Types of stationary space-times in vacuo

In a conventional tensor approach to the classification problem, one may consider the eigenvalue equation of $P^k_i \overset{\text{def}}{=} R^k_i - \frac{1}{3}\delta^k_i R$ (i.e., the trace-free Ricci tensor),

$$P_i^{\ k} v_k = \lambda v_i . \tag{4.1}$$

Though Eq. (4.1) may appear to be unrelated to the geometry of eigenrays, a natural connection between the two emerges in the spinor notation. Let us illustrate this on the important example of the degenerate (type D) class[4] to which all static vacuum metrics and the axisymmetric Papapetrou solutions[12] also belong. For type D fields, two of the eigenvalues in (4.1) are equal so that, from field equations (3.6a) it follows

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*This was first pointed out by R.Penrose.

$$\underline{G} \propto \bar{\underline{G}}. \tag{4.2}$$

Using the canonical decomposition

$$G_{AB} = \alpha_{(A}\beta_{B)} , \tag{4.3}$$

Eq. (4.2) is written

$$\alpha_{(A}\beta_{B)} \propto \alpha^{+}_{(A}\beta^{+}_{B)} . \tag{4.4}$$

Hence we have that the eigenspinors degenerate according to

$$\beta_A = \eta\,\alpha^{+}_{A} . \tag{4.5}$$

For static fields $G_{AB} = \eta\alpha_{(A}\alpha^{+}_{B)}$ is real; $G^{+}_{AB} = -\bar{\eta}\alpha^{+}_{(A}\alpha_{B)} = -\eta\alpha_{(A}\alpha^{+}_{B)}$, so that η is real.

A further specification results from the fairly restrictive condition $\underline{G} = 0$. From field equations (3.6a) it follows that the only stationary vacuum with $\underline{G} = 0$ is the Minkowski space-time*. The weaker condition $\underline{G}\cdot\underline{G} = 0$ offers slightly more generality. Then $\underline{G}$ is a complex null vector so that we may call these fields <u>type null</u> (N). In the canonical decomposition (4.3) the two eigenrays coincide; $G_{AB} = \alpha_{(A}\alpha_{B)}$. Field equations (3.6) imply that <u>the eigenrays of type N fields are geodesic and shear-free.</u> This represents, in three dimensions, the Goldberg-Sachs theorem[11]. With the exception of the Minkowski space-time, a type N field cannot be asymptotically flat since in an asymptotically flat space the leading term in $\underline{G}$ at infinity must be real. We may possibly interpret the type N fields as gravitational standing wave solutions. The plane-fronted standing waves are given by $\rho = 0$ (nondiverging and nonrotating eigenrays). The line elements for type N fields with both $\rho \neq 0$ and $\rho = 0$ are shown in Table I.

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*For the Minkowski space-time we have $R = 0$. The field equations preclude the $R \neq 0$ constant 3-curvature solutions for both vacuum and electrovac fields.

The type N field equations can be written for both line elements as

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \ln P = \frac{1}{8P^2} + \frac{1}{8(\mathrm{Re}\, z)^2}$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\, a = -\frac{1}{4P^2}\, a \, . \qquad \text{(See Table I)} \tag{4.6}$$

The classification scheme can be further refined by considering the propagation properties of the eigenrays in each algebraic class. (Table I.) The Schwarzschild field, being a static solution, is of type D, but it also is exceptional in possessing geodesic and shear-free eigenrays. Its important generalization, the Kerr metric is of type G (general) but still it has geodesic and shear-free eigenrays. The general solution of the class with <u>shearing</u> geodesic eigenrays has been found by Kóta and the author[13]. The resulting line elements do not contain the Kerr solution as a limiting case, rather they are asymptotically nonflat. One of the line elements in this class,

$$ds^2 = \frac{f^0 r^{\gamma^0}}{r^{2\gamma^0} + Q^2}\left(dt - 2\gamma^0 x (f^0)^{-1} Q\, dy\right)^2 - \frac{r^{2\gamma^0} + Q^2}{f^0 r^{\gamma^0}}\left(dr^2 + r^{1-\sigma^0} dx^2 + r^{1+\sigma^0} dy^2\right) \tag{4.7}$$

(f^0, Q, σ^0 and γ^0 are real constants and $\sigma^{0\,2} + \gamma^{0\,2} = 1$) is a type D Papapetrou solution.

The <u>Petrov classification</u> of the Weyl spinor[2] ψ_{ABCD} does not specify the properties of the Killing field for a stationary space-time and as such, tells less about the geometry. It is still worthwile, however, considering the relation between the two schemes. We define a convenient spinor dyad in terms of the normalized eigenspinor ξ_A,

$$o_A = \left(\frac{2}{f}\right)^{1/4} \xi_A \, , \qquad \iota_A = \left(\frac{f}{2}\right)^{1/4} \xi^{+}_A \, . \tag{4.8}$$

Using the notation $G_0 = \underline{G} \cdot \underline{l}$ we find[5]

$$\psi_0 \equiv \psi_{ABCD}\, o^A o^B o^C o^D = -2\sigma G_0 \tag{4.9a}$$

$$\psi_1 \equiv \psi_{ABCD}\, o^A o^B o^C \iota^D = \sqrt{f}\, \kappa G_0 \tag{4.9b}$$

$$\psi_2 \equiv \psi_{ABCD}\, o^A o^B \iota^C \iota^D = f(\rho + G_0) G_0 \, . \tag{4.9c}$$

(4.9a) shows that ℓ lies in one of the principal null directions when $\sigma=0$. For shear-free geodesic eigenrays ($\kappa=\sigma=0$) we have further that $\psi_0=\psi_1=0$, the space-time being algebraically special in the Petrov sense. The eigenspinors of the Kóta-Perjés solutions (with $\sigma\neq 0$) on the other hand, are not proportional to a principal spinor of ψ_{ABCD}. In fact, by transforming to a new spinor basis for which $\psi_0=0$, we find that they are of Petrov type I. Finally, for the null fields we have $G_0=0$ and, from Eq.s (4.9), $\psi_0=\psi_1=\psi_2=0$, so that the Petrov type is III or more special. Unlike in the present scheme, no general statement can be made about the static vacuum fields in the Petrov classification.

5. Electrovac types

Stationary electrovac fields are characterized by the presence of a time-independent, soureeless Maxwell field in the space-time[14]. The Maxwell field is given by a complex 3-vector $\underline{H}$ the real part of which can be interpreted as the electric field and the imaginary part as the magnetic field[15,16]. The field equations can be written

$$(\nabla - \underline{G})\cdot\underline{G} = \overline{\underline{H}}\cdot\underline{H} - \overline{\underline{G}}\cdot\underline{G} \tag{5.1a}$$

$$\nabla\times\underline{G} = \overline{\underline{H}}\times\underline{H} - \overline{\underline{G}}\times\underline{G} \tag{5.1b}$$

$$(\nabla - \underline{G})\cdot\underline{H} = \tfrac{1}{2}(\underline{G} - \overline{\underline{G}})\cdot\underline{H} \tag{5.1c}$$

$$\nabla\times\underline{H} = -\tfrac{1}{2}(\underline{G} + \overline{\underline{G}})\times\underline{H} \tag{5.1d}$$

$$R_{ik} = -G_i\overline{G}_k - \overline{G}_i G_k + H_i\overline{H}_k + \overline{H}_i H_k\ . \tag{5.1e}$$

The appearance of the Maxwell vector $\underline{H}$ in (5.1e) gives rise to a significant complication in the structure of the field. In addition to the eigenrays of the gravitational vector $\underline{G}$ we can define eigenrays of $\underline{H}$. $\underline{G}$ and $\underline{H}$ have a common eigenray congruence when

$$(\underline{G}\times\underline{H})^2 = 0. \tag{5.2}$$

The algebraic properties of $\underline{G}$ and $\underline{H}$ together will determine the structure of the Ricci tensor. A convenient way of revealing the properties which are common to both $\underline{G}$ and $\underline{H}$ is to consider the eigenvalue problem $P_i^k v_k = \lambda v_i$ as the basis of the classification. The picture so obtained will be completed by taking into account the geometric properties of the eigenrays.

The strongest possible condition $P_i^k = 0$ already allows a wide class of solutions. These are the fields with a flat background 3-space. The general solution of the problem has been found by Israel, Wilson[17] and Perjés[16]. The resulting metrics can be interpreted as the external regions in the presence of charged spinning bodies held in equilibrium by the balanced electromagnetic and gravitational forces. The appropriately charged ($e^2 = m^2$) Kerr-Newman solution is a particular member of this class.

The degeneracy condition $\lambda_1 = \lambda_2$ for electrovac fields turns out to be from Eq.(5.1e)

$$(\underline{\bar{G}} \times \underline{G} - \underline{\bar{H}} \times \underline{H})^2 = -4 |\underline{G} \times \underline{H}|^2 . \tag{5.3}$$

For static space-times ($\underline{G}$ real) the Maxwell vector $\underline{H}$ can be made real by a duality rotation. Hence we have that a static electrovac field is of degenerate type provided $\underline{G}$ and $\underline{H}$ have common eigenrays. This further means that the equipotential surfaces f = const. and A_0 = const. (A_0 being the time-like component of the electromagnetic potential) pairwise coincide. Solutions of this type are the axisymmetric Weyl metrics[18] and the Bonnor metric[19] obtained from the Kerr solution(Table 2).

The Kerr-Newman field satisfies (5.2) and its two eigenray congruences are common to both $\underline{G}$ and $\underline{H}$. The two eigenspinors are also the (double) principal spinors of the Weyl curvature. The eigenrays of the Kerr-Newman field are geodesic and shear-free.

One can explicitly solve the equations of electrovac fields with a common geodesic eigenray congruence, even when the shear σ is nonvanishing. The general solution has been found by Lukács and Perjés[8]. Quite similarly to the corresponding vacuum case, the charged Kerr solution cannot be obtained from this class in the shear-free limit $\sigma \to 0$. One of the Lukács-Perjés metrics is singularity-free. This is given by

$$ds^2 = \frac{f^0}{\cos z \cos \bar{z}} \left(dt - \frac{\sigma^0}{2f^0} \operatorname{sh}(2\sigma^0 Q)\, x \right)^2 - \frac{\cos z \cos \bar{z}}{f^0} \left(dr^2 + e^{-2\sigma^0 r} dx^2 + e^{2\sigma^0 r} dy^2 \right);$$

$$\Phi = e^{i\vartheta^0} \sqrt{f^0}\, \operatorname{ch}^{1/2}(2\sigma^0 Q)\, \operatorname{tg} z; \qquad z = \sigma^0 (r + iQ), \tag{5.4}$$

where Φ is the electromagnetic potential and $\sigma^0, Q, f^0, \vartheta^0$ are real constants. The common eigenray congruence is characterized here by $\rho = 0$.

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κ	σ \ P_i^k	O	N null	D degenerate	G general
= 0	= 0	Minkowski	$G=\frac{1}{z+\bar z}\frac{\partial}{\partial z}$ $\quad r=x^1, z=x^2+ix^3$ $l=\frac{\partial}{\partial r}$, $k=-\frac{\partial a}{\partial z}\frac{\partial}{\partial r}+\frac{\partial}{\partial \bar z}$; $a=a(z,\bar z)$; $m=\begin{cases}2Pk & \text{for } \rho=0\\ \frac{2Pk}{r-ia} & \text{for } \rho\neq 0\end{cases}$	Schwarzschild	Kerr ($a \neq 0$)
= 0	≠ 0	Minkowski		Stäckel Kóta-Perjés Papapetrou	Kóta-Perjés
≠ 0		Minkowski			?

Table I.: Types of stationary vacuum

$(\underline{G}\times\underline{H})^2$	κ	σ \ P_i^k	O	D degenerate	G general
= 0	= 0	= 0	Kerr-Newman ($e^2=m^2$) Israel-Wilson-Perjés	Reissner-Nordström	Kerr-Newman
= 0	= 0	≠ 0	Lukács - Perjés Israel-Wilson-Perjés	Lukács - Perjés	Lukács - Perjés
= 0	≠ 0	-	Israel-Wilson-Perjés	Weyl, Bonnor	?
≠ 0	?				

Table II.: Electrovac types

Curriculum vitae

I was born in Budapest, in the year 1943. My parents, both simple workers, are now retired, and my wife Ildikó whom I married in 1971, is a mathematician, scientific co-worker at Institute for Computer Coordination in Budapest. I attended school at the Holy Order of Joseph Calazanc (Piarist), where I studied practical cybernetics and physics.

In the year 1966 I received my Physics Diploma at the Faculty of Natural Sciences of Roland Eötvös University. Following this, I have been working at the Central Research Institute for Physics, in Budapest. The subject of my Diploma was chosen from the theory of nuclear angular correlations, and, accordingly, I worked first at the Nuclear Physics Department. Later on I began to consider problems of general relativity, so I moved on to the Theoretical Division, where I have been working ever since.

In 1969 I had the opportunity of attending the Enrico Fermi School in Italy. This was made possible by a scholarship from the Italian Physical Society. At the GR5 Conference in Tbilisi I lectured on my 3-dimensional approach to time-independent axi-symmetric gravitational fields, which I later published in the journal of Communications in Mathematical Physics. In Copenhagen (at the GR6 Conference) I lectured on an application of the SU(2) spin coefficient method to interacting electromagnetic and gravitational fields. A number of results based on this work have appeared in several papers to follow.

Next, in 1972, I was given the I. Bródy award of the Hungarian Society of Physics for my research into the structure of gravitational equations. In the same year I obtained the academic degree Candidate of Physical Sciences. For the Academic Year 1972-73 I am Leverhulme Visiting Fellow at Birkbeck College, University of London.

Golders Green, on the 27th of March, 1973.