THE NON-LINEAR GRAVITON

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Summary

A new approach to quantized gravitational theory is suggested. It is argued by analogy with Maxwell theory - and also from a principle that (physical) gravitons should carry space-time curvature - that a free graviton should be describable by a complex solution of Einstein's vacuum equations. For a left-handed graviton (pure helicity state $-2\hbar$) we require a solution which is of a particular kind, called right-flat, and which is of positive frequency. A construction is given for obtaining all such solutions, in terms of general curved (top halves of) twistor spaces.

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Despite impressive progress, recently[1], towards the intended goal of a satisfactory quantum theory of gravity, there remain fundamental problems whose solutions do not appear to be yet in sight. In particular, the standard perturbation techniques seem to lead to a quantized Einstein theory which is not renormalizable[1], and it has consequently been argued that Einstein's equations should perhaps be replaced by something more compatible with conventional quantum field theory. There is also the alternative possibility, which has occasionally been aired, that some of the basic principles of quantum mechanics may need to be called into question[2] if we are ever to arrive at a completely satisfactory quantum theory of gravity. It is this second alternative that I wish to explore here; and it is the principle of linear superposition (at least in its standard form) that I shall be calling into question.

Let us first ask: what is the most primitive physical requirement of a quantum theory? As the name implies, what we expect is that it should provide us with <u>quanta</u> - the particles associated with whatever field or fields we are attempting to describe. Thus, a quantum theory of gravity should provide <u>gravitons</u>.

Furthermore, since we are trying to do physics, rather than mathematical exercises, we require that a really physical concept of graviton actually does exist. For the purpose of this essay I am assuming this to be the case. I hope that the somewhat remarkable mathematical result that I shall give lends some independent theoretical support for this supposition.

I shall proceed by analogy with the electromagnetic field, at first. How, indeed, do we provide a mathematical description of a single free photon? I require a description which is both relativistically invariant and gauge invariant. The wave function can then be given as a solution F_{ab} $(= -F_{ba})$ of the free Maxwell equations

$$\nabla_{[a}F_{bc]} = 0 , \qquad \nabla^a F_{ab} = 0 . \tag{1}$$

However, there are two essential differences between classical solutions of the equations (1) and solutions giving wave functions for free photons. In the first place, classical solutions are <u>real</u> while wave functions are <u>complex</u>; in the second (and related to the first), wave functions should be composed only of <u>positive frequencies</u>*. There is also the question of choosing a normalization for the wave function, but I shall not consider that here, as it is not strictly necessary at this stage. But one can go further than this. The description so far is (in our world, where parity conservation is not a general law of nature) a reducible one. We may regard it as describing two <u>different</u> particles simultaneously,

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* There are, of course, many alternative equivalent ways of describing a free photon. The choice I am making here seems to me to be the most natural - and also the only one which fits in with the further ideas I am suggesting.

namely the left-handed and the right-handed photon. To examine this, consider the decomposition of F_{ab} into its self-dual and anti-self-dual parts:

$$F_{ab} = F^{(+)}_{ab} + F^{(-)}_{ab}$$

where

$$F^{(\pm)}_{ab} = \pm i\, e_{abcd}\, F^{(\pm)cd} ,$$

e_{abcd} being the alternating tensor. If F_{ab} is real, $F^{(+)}_{ab}$ and $F^{(-)}_{ab}$ are complex conjugates of one another, but for a wave function, they are simply independent complex quantities, each of positive frequency (and therefore necessarily not complex conjugates of one another). In fact $F^{(+)}_{ab}$ describes a left-handed photon (helicity $-\hbar$) and $F^{(-)}_{ab}$ a right-handed photon (helicity $+\hbar$) .

A spinor description can also be used. Put

$$F_{ab} = \phi_{AB}\, \varepsilon_{A'B'} + \varepsilon_{AB}\, \tilde{\phi}_{A'B'} \qquad (2)$$

(the abstract index notation being employed, for convenience), where

$$F^{(+)}_{ab} = \phi_{AB}\, \varepsilon_{A'B'} , \qquad F^{(-)}_{ab} = \varepsilon_{AB}\, \tilde{\phi}_{A'B'} .$$

Then for a classical (real) field we have $\tilde{\phi}_{A'B'} = \bar{\phi}_{A'B'}$ whereas for a photon wave function ϕ_{AB} and $\tilde{\phi}_{A'B'}$ are independent (symmetric) spinors describing the left-handed and right-handed components of the photon[4]. The Maxwell equations (1) become

$$\nabla^{AA'}\phi_{AB} = 0 , \qquad \nabla^{AA'}\tilde{\phi}_{A'B'} = 0 . \qquad (3)$$

To describe a pure $-\hbar$ helicity state we must take $\tilde{\phi}_{A'B'} = 0$; for a pure $+\hbar$ helicity state we take $\phi_{AB} = 0$. The requirement that ϕ_{AB} and $\tilde{\phi}_{A'B'}$ have positive frequency may be stated in the following geometrical way: it is possible to extend the domain of definition of each field quantity holomorphically to points of complexified Minkowski space $\mathbb{C}\mathbb{M}$, whose position vectors $x^a + i\, y^a$ have imaginary part y^a lying on or within the past null cone.(3) This portion of $\mathbb{C}\mathbb{M}$ will be denoted: $\mathbb{C}\mathbb{M}^+$.

Let us now turn to the case of gravity. There is an obvious analogue to the above description, where two four-index symmetric spinors ϕ_{ABCD}, $\tilde{\phi}_{A'B'C'D'}$ are employed in a Minkowski background space to provide a (complex) linearized curvature tensor

$$K_{abcd} = \phi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \varepsilon_{AB}\, \varepsilon_{CD}\, \tilde{\phi}_{A'B'C'D'} \, .$$

The discussion is exactly as above. A common view of a graviton has, indeed, been essentially this, namely something which refers to weak-field gravitational perturbations of a space-time (normally flat). However, it is at this point that I wish to take issue with the standard view. I cannot (now) believe that a physical graviton should be described by linear gravitational theory. My reasoning is as follows. If we start with flat space, and then add one (linearized) "graviton", the space remains flat. The null cones have not shifted. We add a second such "graviton" and a third and a fourth, and the space is still flat with null cones still locked in their original Minkowskian positions. With such a perturbative viewpoint it is only after an infinite number of "gravitons" have been added that the space can become curved. The situation may be compared with a power series expansion. For any finite number of terms the function

$$1 + \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^n}$$

has a pole singularity locked at the position $z = 0$. But the sum to infinity

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots = \frac{z}{z-1}$$

has its pole shifted to the position $z = 1$. The usual perturbative expansions of graviton propagators have a similar behaviour, with poles corresponding to the Minkowskian null cones right up until the sum to infinity has been performed. Surely, if it has meaning to talk about physical gravitons, it should have meaning to talk about finite numbers of such gravitons. And if Einstein's theory is physically appropriate, then surely each graviton itself carries its measure of curvature. I cannot believe that curvature emerges only after an infinite sum has been performed.

So rather than working with linear theory, let us attempt to apply the electromagnetic analogy instead to the full non-linear Einstein theory. Indeed, a decomposition of the Riemann curvature of a vacuum space-time can be expressed in a form analogous to (2):

$$R_{abcd} = \Psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'} + \varepsilon_{AB}\,\varepsilon_{CD}\,\tilde{\Psi}_{A'B'C'D'} ;$$

where the two terms on the right are the self-dual and anti-self-dual parts of the vacuum curvature, respectively. For a classical (real) space-time, $\tilde{\Psi}_{A'B'C'D'}$ must be the complex conjugate of Ψ_{ABCD}. But to describe a (non-linear) graviton we must expect, by analogy with the electromagnetic case, that Ψ_{ABCD} and $\tilde{\Psi}_{A'B'C'D'}$ will become independent quantities each, in some appropriate sense, of positive frequency. The curvature R_{abcd} becomes complex, so we are dealing with some form of complex Riemannian space. The metric g_{ab} must, indeed, be complex symmetric (holomorphic, not Kähler). The complex Ricci tensor must vanish. The complex vacuum Bianchi identity becomes

$$\nabla^{AA'} \Psi_{ABCD} = 0 , \quad \nabla^{AA'} \tilde{\Psi}_{A'B'C'D'} = 0$$

in close analogy with (3) .

But if such a situation describes a graviton it would generally be a graviton of mixed helicities. For a pure graviton in a helicity state we would apparently require $\tilde{\Psi}_{A'B'C'D'} = 0$ (helicity $-2\hbar$) or else $\Psi_{ABCD} = 0$ (helicity $+2\hbar$). I shall refer to a complex vacuum space-time for which $\tilde{\Psi}_{A'B'C'D'} = 0$ as <u>right-flat</u>; and one for which $\Psi_{ABCD} = 0$ as <u>left-flat</u>. Thus, I am proposing that a single left-handed graviton is described by a positive-frequency right-flat (left-curved) complex space-time; and a right-handed graviton by a left-flat space-time*.

Such a description may seem somewhat mysterious (and "positive-frequency" has yet to be clearly defined in this context). But it is, to my mind, a very remarkable fact that the general such complex space-time can be constructed <u>automatically</u> by way of <u>twistor theory</u>[4],[1]! There are already various reasons for believing that twistor theory may be able to provide some new insights into the workings of nature; and here we have an apparently independent role for this theory. It will not be possible to go into any detail, but the essential mathematical result can be briefly described.

Let us first recall[4] how complex (compactified) Minkowski space $\mathbb{CM}$ arises from standard (flat) twistor space $\mathbb{T}$. The

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* The concepts of right-flat and left-flat space-times arose historically from E. T. Newman's construction of the spaces[5] referred to as "Heavens". Though much of this essay has its origins in Newman's construction, I shall not have need to refer to it explicitly here.

space $\mathbb{T}$ is a complex four-dimensional vector space endowed with a Hermitian form Σ of signature (++--) . The parts of $\mathbb{T}$ at which Σ is positive, negative, or zero are labelled $\mathbb{T}^+$, $\mathbb{T}^-$, $\mathbb{N}$, respectively. The complex two-dimensional linear subspaces of $\mathbb{T}$ correspond to the points of $\mathbb{CM}$; those lying in $\mathbb{N}$ define the points of real (compactified) Minkowski space $\mathbb{M}$; those lying in $\mathbb{T}^+ \cup \mathbb{N}$ define the points of $\mathbb{CM}^+$. Often it is convenient to consider the projective twistor space $\mathbb{PT}$ (where the prefix $\mathbb{P}$ means "projective"). The points of $\mathbb{CM}$ are then represented in $\mathbb{PT}$ by complex projective lines (compact holomorphic curves having topology S^2); the points of $\mathbb{M}$ by projective lines in $\mathbb{PN}$ and those of $\mathbb{CM}^+$ by projective lines in $\mathbb{PT}^+ \cup \mathbb{PN}$. The space $\mathbb{T}$ is also endowed with a (Poincaré invariant) 1-form and 2-form:

$$\underset{\sim}{\iota} = I_{\alpha\beta} Z^\alpha dZ^\beta, \quad \underline{\tau} = d\underset{\sim}{\iota} = I_{\alpha\beta} dZ^\alpha \wedge dZ^\beta$$

(the form $\underline{\tau}$ being degenerate, of rank 2); and with (a conformally invariant) 3-form and 4-form:

$$\underline{\rho} = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$$

$$\underline{\omega} = d\underline{\rho} = \varepsilon_{\alpha\beta\gamma\delta}\, dZ^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \quad ,$$

in standard twistor notation[1].

To construct a left-handed non-linear graviton we proceed as follows. We consider the bounded portion $\mathbb{PT}^+ \cup \mathbb{PN}$ of projective twistor space, deleting the part $\mathbb{PT}^-$. We then perturb its complex structure in an arbitrary fashion to obtain a new space $\mathcal{J}$. By appealing to a theorm of Kodaira[6] (and employing some observations due to M. F. Atiyah) we can infer the existence of a complex four-parameter family of compact holomorphic curves in $\mathcal{J}$, which are the analogues of the projective lines in $\mathbb{PT}^+$.

Each such curve now defines a point in an abstract complex four-manifold $\mathcal{M}$. The space $\mathcal{M}$ naturally acquires a (complex) conformal structure, where null separation in $\mathcal{M}$ corresponds to intersection of the corresponding curves in $\mathcal{T}$. This conformal structure can be shown to be Riemannian (by general theory) with a right-flat conformal curvature. We now require that the forms $\underline{\iota}$, $\underline{\tau}$, $\underline{\rho}$, $\underline{\omega}$ have been introduced, with the same stated properties as for flat twistors, but otherwise arbitrary, We apparently need solve no equations to achieve all this. But, remarkably, the resulting complex four-space $\mathcal{M}$ acquires the structure of a general right-flat solution of the Einstein vacuum equations!

To ensure that $\mathcal{M}$ is, in some appropriate sense, of positive frequency, we require that $\mathcal{T}$ is sufficiently extended that it is validly analogous to $\mathbb{PT}^+ \cup \mathbb{PN}$. I cannot explain the details here, but the requirement seems to be that the boundary of $\mathcal{T}$ is ruled appropriately by a real three-parameter family of holomorphic curves. Again, it seems that no equations need be solved to achieve this. Details will be published elsewhere.

There is clearly much more that needs to be done for this line of approach to quantum gravity to be made the basis of a physical theory. We require a definition of scalar product between these one-particle states. And how are we to build up a many-graviton theory from this one-graviton approach*? Further work is in progress.

My thanks are especially due to E. T. Newman and M. F. Atiyah for supplying vital ideas.

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* A modified form of Geroch's proposal[7] seems worth exploring here. I am grateful to A. Ashtekar for suggesting this.

References

(1) C. J. Isham, R. Penrose and D. W. Sciama (Eds.), Quantum Gravity: An Oxford Symposium (Oxford Univ. Press, 1975).

(2) B. Mielnik, Commun. Math. Phys. **37** (1974) 221.

(3) R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That (W. A. Benjamin, Inc., New York, 1964).

(4) R. Penrose and M.A.H. MacCallum, Physics Reports **6C** (1973) 242.

(5) E. T. Newman, report to Tel Aviv Conference 1974.

(6) K. Kodaira, Amer. J. Math. **85** (1963) 79.

(7) R. P. Geroch, Comm. Math. Phys. **26** (1972) 271.

Résumé

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- Proc. Camb. Phil, Soc. 72 (Nov. 1972).
- Proc. Camb. Phil. Soc. 73 (Jan. 1973).
- Phys. Rev. D. 8, 2369 (1973)

- Matrix and Tensor Quarterly, 24, No. 3 (1974).
- Phys. Rev. Lett., 33, 1461 (1974).
- Journal of Math. Phys. to Appear March (1975).
- Journal of Math. Phys. to Appear April (1975).
- Journal of Computational Physics to Appear (1975).

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