

Biographical Sketch

Leonard Parker received his Ph.D. in 1967 from Harvard University. He works on problems involving quantum theory and gravity. He is the author of more than forty research papers, and is best known for his work on the creation of elementary particles by strong gravitational fields. He holds the position of Professor at the University of Wisconsin-Milwaukee.

Summary

A one-electron atom is considered in a general curved spacetime. The Hamiltonian of the Dirac equation is written in Fermi normal coordinates, including all interaction terms of first order in the Riemann tensor of the spacetime. Expressions are obtained for the shifts in various atomic energy levels caused by the curvature. There is a possibility that these shifts would be observable in the spectrum of Hydrogen falling into small black holes (radius about 10^{-3} cm) left over from the early universe.

The Atom As A Probe of Curved Spacetime*

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An atom placed in a strong gravitational field will be distorted by its interaction with the local curvature of spacetime. As a result of this interaction, a nearby observer at rest with respect to the atom would see a change in its spectrum. The shift in energy of each atomic level would depend on the Riemann curvature tensor at the position of atom. These shifts would not be the same for all energy levels. Thus, in principle, the atomic spectrum carries unambiguous information about the local curvature at the position of the atom. This effect would be appreciable only in regions of large curvature. For example, one finds that for the energy level shifts in Hydrogen to be of the order of the Lamb shift (4.4×10^{-6} eV), the characteristic radius of curvature of spacetime at the position of the atom would have to be about 10^{-3} cm.

An observer far from the atom would see additional cosmological, gravitational, and Doppler shifts of the spectral lines. These shifts would be the same for all spectral lines, and could thus be separated from the shifts produced by the curvature at the position of the atom. Hence, from the observed spectrum, one could deduce the shifts in the proper energy levels of the atom caused by the local curvature. I have recently undertaken the calculation of these energy level shifts for a one-electron atom in an arbitrary curved spacetime. In this essay, I will describe the calculation and give the results for a number of energy levels.

The calculation of the energy levels of a one-electron atom in curved spacetime is a fundamental problem, which involves interesting complications of a theoretical nature. There are contradictory conclusions in the literature, and to my knowledge no one has given explicit expressions for the energy level shifts, as I do here. A critique of the previous literature, including references, is given by Audretsch and Schäfer, who considered the Hydrogen atom in certain cosmological metrics [1].

I assume that the one-electron atom is governed by the Dirac equation generalized to curved spacetime. That equation is [2,3]

$$(\underline{\gamma}^\mu(x)\nabla_\mu + m)\psi(x) = 0 \quad (1)$$

where

$$\underline{\gamma}^\mu(x)\underline{\gamma}^\nu(x) + \underline{\gamma}^\nu(x)\underline{\gamma}^\mu(x) = 2g^{\mu\nu}(x) , \quad (2)$$

and ∇_μ denotes the covariant derivative acting on the four component spinor field $\psi(x)$. Minimal coupling to the electromagnetic field is included in the definition of ∇_μ . To good approximation, one can take the atom to be in free fall along a geodesic, G , of the spacetime during the time required for an atomic transition. It is convenient to use coordinates which are locally inertial or normal at the position of the atom. The most appropriate set of normal coordinates for dealing with a problem involving energy levels are Fermi normal coordinates [4]. They are normal along the path of the atom, and therefore remain normal during the entire emission process.

In Fermi normal coordinates, each spacelike hypersurface of constant x^0 is generated by the set of spacelike geodesics normal at a point to the time-like geodesic G along which the atom is falling. The time x^0 of an event in the hypersurface is the proper time along G at the point where it intersects the hypersurface. For an arbitrary spacetime, the metric in these coordinates takes the following form, to first order in the Riemann tensor [5,6]:

$$g_{00} = -1 - R_{0l0m} x^l x^m, \quad g_{0i} = -\frac{2}{3} R_{0lim} x^l x^m, \quad g_{ij} = \delta_{ij} - \frac{1}{3} R_{iljm} x^l x^m, \quad (3)$$

where Latin indices range from 1 to 3, and $R_{\mu\nu\lambda\sigma}$ is evaluated at the center of mass or energy of the atom, which is taken to be the spatial origin. Thus $R_{\mu\nu\lambda\sigma}$ is a function of x^0 . I assume that on an atomic time scale the time rate of change of the Riemann tensor, as measured along the spacetime path of the atom, is sufficiently slow that time derivatives of $R_{\mu\nu\lambda\sigma}$ can be neglected. This condition is necessary for well defined energy levels to exist.

The Dirac equation can be put into the Hamiltonian form (with $\hbar = c = 1$):

$$i\partial_0 \psi = H\psi, \quad (4)$$

with

$$H = -i(g^{00})^{-1} \underline{\gamma}^0 \underline{\gamma}^i (\partial_i - \Gamma_i) + i\Gamma_0 - i(g^{00})^{-1} \underline{\gamma}^0 m, \quad (5)$$

where the Γ_μ are the spinor affine connections, including a term $-ieA_\mu$, where e is the magnitude of the electron charge and A_μ is the electromagnetic vector potential. After a very long calculation, I find that in Fermi normal coordinates the Hamiltonian of the one-electron atom is

$$H = H_0 + H_I, \quad (6)$$

with

$$H_0 = -i\alpha^i \partial_i + m\beta - \zeta r^{-1}, \quad (7)$$

and

$$\begin{aligned} H_I = & -\frac{i}{2} R_{olom} x^\ell x^m \alpha^i \partial_i - \frac{i}{6} R_{iljm} x^\ell x^m \alpha^j \partial^i \\ & - \frac{i}{6} R_{oljm} x^\ell x^m \alpha^j \alpha^i \partial_i - \frac{i}{2} R_{olim} x^\ell x^m \partial^i \\ & + \frac{i}{4} \alpha^j (R_{jm} - R_{ojom}) x^m + \frac{i}{4} \alpha^i \alpha^j R_{oimj} x^m \\ & + \frac{1}{6} \zeta R_{olmi} x^\ell x^m \alpha^i r^{-1} + \frac{1}{2} \zeta R_{oi} \alpha^i r \\ & - \frac{1}{12} \zeta (R_{\ell m} + 3R_{olom}) x^\ell x^m r^{-1} + \frac{1}{12} \zeta (R + 4R_{oo}) r \\ & + \frac{1}{2} m R_{olom} x^\ell x^m \beta - \frac{1}{6} m R_{ilom} x^\ell x^m \beta \alpha^i, \end{aligned} \quad (8)$$

where $\zeta = Ze^2$ (Z is the atomic number of the nucleus), m is the reduced mass, and the α^i and β are the standard Dirac matrices. This result contains all terms of first order in the curvature tensor, including the corrections to the electromagnetic field.

One can check by direct calculation that H of Eq. (6) is Hermitian with

respect to the conserved scalar product

$$(\phi, \psi) \equiv -i \int d^3 \mathbf{x} \sqrt{-g} \phi^\dagger \beta \underline{\gamma}^0(\mathbf{x}) \psi . \quad (9)$$

This Hermiticity implies that the eigenvalues of H are real. On the other hand, neither H_0 nor H_I is Hermitian with respect to the scalar product of Eq. (9). Therefore, one must exercise care in developing the perturbation theory of stationary states (including degeneracy) based on the flat spacetime relativistic Hamiltonian H_0 . Rather than go into the details here, let me summarize the results to date.

Using the known exact spinor solutions [7] of the eigenvalue problem of H_0 , I find after lengthy calculation that, to first order in the curvature, both of the degenerate $1S_{\frac{1}{2}}$ levels are shifted by the same amount:

$$E^{(1)}(1S_{\frac{1}{2}}) = \frac{1}{12} \zeta^{-2} \gamma(\gamma+1)(2\gamma+1)m^{-1}R_{00} + \frac{1}{72} (2\gamma+1)m^{-1}(3R+4R_{00}) , \quad (10)$$

where

$$\gamma = (1-\zeta^2)^{\frac{1}{2}} , \quad \zeta = Ze^2 . \quad (11)$$

This includes all terms linear in $R_{\alpha\beta\gamma\delta}$ and is valid to all orders in ζ . To higher order in the curvature, the $1S_{\frac{1}{2}}$ levels will be shifted by different amounts.

The $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ eigenstates of H_0 are four-fold degenerate. One finds that, to lowest order in ζ , the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states are not mixed by the perturbation H_I . The energy of the $2S_{\frac{1}{2}}$ states is separated from that of the $2P_{\frac{1}{2}}$ states, and each of the resulting levels is two-fold degenerate. The energy shifts of those states, to lowest order in ζ , are

$$E^{(1)}(2S_{\frac{1}{2}}) = 7\zeta^{-2} m^{-1} R_{00} \quad (12)$$

and

$$E^{(1)}(2P_{\frac{1}{2}}) = 5\zeta^{-2} m^{-1} R_{00} . \quad (13)$$

It follows that the splitting of the $2S_{1/2}$ and $2P_{1/2}$ levels in Hydrogen caused by the curvature will be of the same order as that caused by the Lamb shift (4.4×10^{-6} eV) when the characteristic radius of curvature, D , of the space-time is about 10^{-3} cm. The above results are valid when the energy shifts caused by the curvature are smaller than the relativistic fine structure, as would be the case in most gravitational fields.

When the curvature is sufficiently great ($D < 10^{-4}$ cm) that the energy shifts are larger than the relativistic fine structure, one can work in the non-relativistic limit. In the non-relativistic regime, I find the following shifts for the 1S, 2S, and 2P levels:

$$\begin{aligned}
 E^{(1)}(1S) &= \frac{1}{2} \zeta^{-2} m^{-1} R_{oo} , & E^{(1)}(2S) &= 7\zeta^{-2} m^{-1} R_{oo} , \\
 E_1^{(1)}(2P) &= 3\zeta^{-2} m^{-1} (R_{oo} + 2R_{oxox}) , & E_2^{(1)}(2P) &= 3\zeta^{-2} m^{-1} (R_{oo} + 2R_{oyoy}) , \\
 E_3^{(1)}(2P) &= 3\zeta^{-2} m^{-1} (R_{oo} + 2R_{ozoz}) , & & (14)
 \end{aligned}$$

where the x, y, and z directions of the normal coordinates have been chosen such that R_{oioj} is diagonal at the origin. For example, in the Schwarzschild metric with the x-axis of the normal coordinates in the radial direction, one has [5] $R_{oxox} = 2Mr_s^{-3}$, $R_{oyoy} = R_{ozoz} = -Mr_s^{-3}$, where M is the Schwarzschild mass and r_s is the value of the Schwarzschild radial coordinate at the position of the atom. In future work, these results will be extended in both the non-relativistic and relativistic regimes.

The energy shifts, $E^{(1)}$, obtained above are those which would be measured by a detector located near the atom and at rest relative to the atom. A distant observer could, in principle, deduce these shifts, $E^{(1)}$, from the observed spectrum because the $E^{(1)}$ are not the same for all energy levels. The calculated values of the $E^{(1)}$ are independent of the theory by which the curvature tensor $R_{\mu\nu\lambda\sigma}$ of the spacetime is generated. Within the context of general

relativity, one may possibly be able to observe these effects in the spectrum of Hydrogen falling into a small black hole (radius about 10^{-3} cm) left over from the early stages of the universe. Heating of the infalling gas, and the intrinsic temperature of the black hole will produce a thermal broadening of the spectral lines, which will limit the resolution of the observations. Nevertheless, there is a possibility that these shifts can be observed, in which case they would serve to identify such black holes.

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