PLANCK LENGTH IS THE LOWER BOUND TO ALL PHYSICAL LENGTH SCALES*

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Abstract

The effect of quantum fluctuations of gravity on the measurement of proper distances is considered. It is shown that, when the length scales are of the order of Planck length, the concept of a unique distance between points ceases to exist. It is also shown that the quantum expectation value of the proper length is bounded from below by Planck length in any space-time.

1. Flat space as gravitational vacuum

Classical general relativity identifies gravity with space-time curvature. In this picture the proper (physical) distance between two events x^i and $x^i + dx^i$ is given by,

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k \tag{1}$$

where g_{ik} is determined by the Einstein's equations. In the absence of gravitational field, g_{ik} assumes the familiar flat space value η_{ik} = dia (1, -1, -1, -1) and we get,

$$ds^2 = \eta_{ik}\, dx^i\, dx^k = dt^2 - dx^2 - dy^2 - dz^2. \tag{2}$$

It is tacitly assumed in the classical theory that g_{ik} at any single event x^i can be measured to arbitrary level of accuracy. Thus the proper lengths in (1) and (2) can be determined as accurately as one wants. This immediately leads to the conclusion that,

$$\underset{x^i \to y^i}{\mathrm{Lt}} \; ds^2 = 0 \tag{3}$$

where $y^i = x^i + dx^i$. This, rather trivial, result states that the proper interval goes to zero as the events approach each other.

Classical gravity, however, is only an approximation to quantum gravity. The flat space-time should be more properly considered to be the vacuum state of quantum gravity. The metric tensor g_{ik} becomes a quantum field and is bedevilled by the quantum fluctuations. Even in the flat space-time the vacuum fluctuations of gravity will be present. Thus, it is no longer possible to measure the value of g_{ik} <u>at a single event</u> x^i and obtain a unique value for the space-time interval in (1) or (2). A more detailed, probabilistic description is required.

In particular, we expect the quantum fluctuations to grow very large at small distances. Therefore, it is not clear how the result in (3) would be modified when the fluctuations in the metric tensor are taken into account. We shall show below how these questions can be settled in a simple model for quantum gravity.

2. Quantum conformal fluctuations

Considerable progress can be made in discussing the quantum dynamics of gravitational field, if the attention is confined to the conformal degree of freedom of gravity //(1,2,3)//. Quantum gravity can be approached through the path integral,

$$K = \int \mathcal{D} g_{ik} \exp \; iS \, [g_{ik}] . \tag{4}$$

where,

$$S = \frac{1}{16\pi G} \int R \sqrt{-g}\, d^4x \quad \equiv \frac{1}{12L_p^2} \int R \sqrt{-g}\, d^4x \,. \tag{5}$$

Most of the contributions to the path integral are expected to arise from the classical solution $g_{ik} = \bar{g}_{ik}$ (say). In considering the quantum conformal fluctuations, one evaluates the path integral in (4) over a class of metrics which are conformal to $\bar{g}_{ik}$: i.e. we take

$$g_{ik} = [\, 1 + \phi(x)]^2\, \bar{g}_{ik}. \tag{6}$$

In terms of ϕ , the path integral becomes,

$$K = \int \mathcal{D}\phi \exp\{- \frac{i}{2L_p^2} \int [\phi^i \phi_i - \frac{1}{6} \bar{R}(1 + \phi)^2] \sqrt{-\bar{g}}\, d^4x\} \tag{7}$$

which can be evaluated in closed form due to the quadratic nature of the ϕ dependence. Detailed discussion of this approach, its relevance to quantum gravity etc. can be found in references cited previously //(1,2,3)//and will not be repeated here.

In particular, the above formalism can be used to answer the following question : What is the probability that the conformal fluctuations has a given value $\phi(\underset{\sim}{x})$ in the gravitational vacuum (flat space)? The answer is given by,

$$\mathcal{P}\,[\phi(\underset{\sim}{x})] = N \exp\left(-2\int \frac{d^3\underset{\sim}{k}}{(2\pi)^3}\; |\underset{\sim}{k}|\; |q_{\underset{\sim}{k}}|^2\right)$$

$$= N \exp\left(-\frac{1}{4\pi^2 L_P^2}\int d^3\underset{\sim}{x}\; d^3\underset{\sim}{y}\; \frac{\nabla\phi(\underset{\sim}{x})\cdot\nabla\phi(\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|^2}\right) \qquad (8)$$

where

$$\phi(\underset{\sim}{x}) = \int q_{\underset{\sim}{k}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, \frac{d^3\underset{\sim}{k}}{(2\pi)^3}\,. \qquad (9)$$

The expression (8) denotes the square of the 'ground state wave functional' for gravity and is derived and discussed in references //(4,5)//.The time independence of $\mathcal{P}$ reflects the fact that ground state is a stationary state [For the corresponding expression in electrodynamics see //(6,7)//]. The vacuum fluctuations of gravity can be studied using this probability functional.

3. Quantum fluctuations and length measurements.

Let us see how the fluctuations of the conformal factor affect the measurement of proper length between points in space $\underset{\sim}{x}$ and $\underset{\sim}{y}$ (at time t) in flat space. To do this, one has to make a measurement of the fluctuating field $\phi(x)$ in (7). Consider an experiment which achieves this measurement with a spatial resolution of L (say). In other words, the measurement cannot distinguish points $\underset{\sim}{x}$ and $\underset{\sim}{y}$ as distinct if

$|\underset{\sim}{x} - \underset{\sim}{y}| < L$. (An ideal experiment, of course, is a special case of $L = 0$). If the sensitivity profile of the set up is denoted by $f(\underset{\sim}{r})$ then, we will actually be measuring the field $\phi(\underset{\sim}{x})$ "coarse-grained" over the scale L :

$$\phi_f(\underset{\sim}{x}) \equiv \int \phi(\underset{\sim}{x} + \underset{\sim}{r})\, f(\underset{\sim}{r})\, d^3\underset{\sim}{r} \ . \tag{10}$$

(The function f is taken to be zero for $|\underset{\sim}{r}| > L$ and is of the order of unity for $|\underset{\sim}{r}| < L$. Thus the experiment does not distinguish between points $\underset{\sim}{x}$ and $\underset{\sim}{x} + \underset{\sim}{r}$ when $|\underset{\sim}{r}| < L$.) The quantity of interest, is the probability that ϕ_f has a particular value η (say). Since the probability distribution for $\phi(x)$ is given by (8), this can be easily seen to be given by,

$$\mathcal{A}[\phi_f = \eta] = \int \mathcal{D}\phi(\underset{\sim}{x})\, \delta(\phi_f - \eta)\, \mathcal{P}[\phi(\underset{\sim}{x})]$$

$$= N\int_{-\infty}^{+\infty}\frac{d\lambda}{2\pi} \int \mathcal{D}\,\phi(\underset{\sim}{x}) \exp i\lambda\, [\phi_f - \eta]$$

$$\times \exp\left(-\frac{1}{4\pi^2 L_p^2} \int d^3\underset{\sim}{x}\, d^3\underset{\sim}{y}\, \frac{\nabla\phi\cdot\nabla\phi}{|\underset{\sim}{x} - \underset{\sim}{y}|^2}\right).$$

Performing the integrations (for details, see ref.8), we get,

$$\mathcal{A}[\eta] = \left(\frac{1}{2\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{\eta^2}{2\Delta^2}\right) \tag{12}$$

with,

$$\Delta^2 = L_p^2 \int \frac{d^3k}{(2\pi)^3} \frac{|f(\underset{\sim}{k})|^2}{|\underset{\sim}{k}|} . \qquad (13)$$

Suppose we take $f(r)$ to be the gaussian sensitivity profile,

$$f(\underset{\sim}{r}) = (\frac{1}{2\pi L^2})^{3/2} \exp(-\frac{|\underset{\sim}{r}|^2}{2L^2}) \qquad (14)$$

then, from (13) we get,

$$\Delta^2 = L_p^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\underset{\sim}{k}|} |f(\underset{\sim}{k})|^2 = \frac{L_p^2}{L^2} . \qquad (15)$$

For any other choice of $f(\underset{\sim}{r})$ with a characteristic width of L, the answer will be of the same order : $\Delta^2 \sim L_p^2/L^2$.

Thus as long as one confines oneself to length measurements averaged over many Planck lengths, (i.e. for $L >> L_p$), Δ is almost zero and the probability in (12) is sharply peaked at $\eta = 0$. The physical distance between the two events $(\underset{\sim}{x},\underset{\sim}{y})$ is hardly affected by quantum fluctuations. However as the accuracy of measurement increases $(L \rightarrow 0)$, the dispersion in quantum fluctuations grow and length determinations become fuzzy. Using (12) and (6), one can show that the probability that two points $(\underset{\sim}{x},\underset{\sim}{y})$ are separated by a proper distance R is given by,

$$P(R) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{ - \frac{(R - R_o)^2}{2\sigma^2} \right\} \qquad (16)$$

where,

$$R_o = |\underset{\sim}{x} - \underset{\sim}{y}| \; ; \qquad \sigma^2 = R_o^2 \left(\frac{L_p^2}{L^2}\right) . \qquad (17)$$

To have a well defined concept of length between two points, one must have $\sigma^2 << R_o^2$ implying $L >> L_p$. As the measurement becomes more and more accurate, we can only talk about the probability for a particular value for the length. The concept of definite proper length breaks down at $L \cong L_p$. Equation (16) is the main result of this analysis.

4. Lower bound to proper length

Classically nothing prevents one from considering two events that are arbitrarily close; the proper length tends to zero in this limit. However it is physically meaningless to talk about distances below the resolution limit L. If this resolution limit is taken to zero, then equation (16) predicts infinite uncertainty in the proper length.

Instead of considering the fluctuations in the conformal factor, one may look at the expectation value of the line interval

$$<0|ds^2|0> \equiv < g_{ik}(x) > dx^i\, dx^k = (1 + <\phi^2(x)>)\, dx^i\, dx^k\, \bar{g}_{ik}. \qquad (18)$$

However, it is well known that $<\phi^2>$ diverges for quantum fields. Also we notice that ds^2 involves for it definition two events x^i and $y^i \equiv x^i + dx^i$. It is better to consider $<\phi^2(x)>$ as the limit,

$$< \phi^2(x) > = \underset{x \to y}{\mathrm{Lt}} < \phi(x)\phi(y)>. \qquad (19)$$

In flat space, the limiting distance between two space points $\underset{\sim}{x}$ and $\underset{\sim}{y}$ (at some time t, say) is given by,

$$\underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}}\ \ell^2(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}}\ |\underset{\sim}{x} - \underset{\sim}{y}|^2 = 0 \qquad (20)$$

in the classical limit. When quantum fluctuations are included, this is replaced by,

$$\underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}} < \ell^2(\underset{\sim}{x},\underset{\sim}{y})> = \underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}}\ (1 + <\phi(\underset{\sim}{x},t)\phi(\underset{\sim}{y},t)>)\ |\underset{\sim}{x} - \underset{\sim}{y}|^2 \qquad (21)$$

The expectation value can be evaluated by standard field theory techniques and is given by, (see ref. 8)

$$< \phi(\underset{\sim}{x},t)\ \phi(\underset{\sim}{y},t)> = \frac{4\pi^2 L_p^2}{|\underset{\sim}{x} - \underset{\sim}{y}|^2}. \qquad (22)$$

Therefore,

$$\underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}} < \ell^2(\underset{\sim}{x},\underset{\sim}{y}) > = \underset{\underset{\sim}{x} \to \underset{\sim}{y}}{\mathrm{Lt}} \left(1 + \frac{4\pi^2 L_p^2}{|\underset{\sim}{x} - \underset{\sim}{y}|^2}\right) |\underset{\sim}{x} - \underset{\sim}{y}|^2 \qquad (23)$$

$$= 4\pi^2 L_p^2$$

In other words, the expectation value of the proper length between two events is bounded at Planck length! This is another simple conclusion that follows from the study of quantum conformal fluctuations.

This result is far more general than indicated by the derivation above. First of all, for any two events x^i and $x^i + \epsilon^i$, in flat space, the expectation value has the form,

$$< 0 \mid \phi(x + \epsilon) \; \phi(x) \mid 0 > = - \frac{4\pi^2 L_p^2}{(\epsilon^i \epsilon_i)} . \qquad (24)$$

Thus,

$$\underset{\epsilon \to 0}{\mathrm{Lt}} < \ell^2(x, x+\epsilon) > = \underset{\epsilon \to 0}{\mathrm{Lt}} \left(1 - \frac{4\pi^2 L_p^2}{\epsilon^i \epsilon_i}\right) \epsilon^i \epsilon_i = -4\pi^2 L_p^2 . \qquad (25)$$

[The minus sign shows that the lower bound arises from the limiting value of spacelike separations]. The result can also be generalized to arbitrary curved space-time because of the following fact : In any space-time,

$$\underset{x \to y}{\mathrm{Lt}} < \phi(x)\phi(y) > = -\frac{4\pi^2 L_p^2}{s^2} \tag{26}$$

where s^2 is proper interval between x and y (see e.g. ref. 9). It is clear that the analysis can be carried over to any space-time.

This result has important implications for the ultraviolet divergences which arise in the quantum theory of fields. Consider the Green's function for a massless free scalar field $\psi(x)$ in flat space. It is usually taken to be, [$\bar{S}$ is the action for the scalar field],

$$G_o(x,y) \equiv \int \mathcal{D}\psi(x) \;\; \psi \;\; (x)\psi(y) \exp \frac{i}{2} \bar{S} [\psi]. \tag{27}$$

However, we have just seen that even flat space undergoes vacuum fluctuations of gravity. Thus one should average $G_o(x,y)$ over the fluctuations of the metric tensor, obtaining,

$$G(x,y) \equiv < G_o(x,y) > \equiv \int \mathcal{D} \phi(x) \int \mathcal{D}\psi(x) \exp i(\bar{S} + S_g). \tag{28}$$

Since

$$G_o(x,y) = \frac{4\pi^2}{(x - y)^2} \tag{29}$$

we get,

$$G(x,y) \cong \frac{4\pi^2}{\langle (x-y)^2 \rangle} = \frac{4\pi^2}{(x-y)^2 - 4\pi^2 L_p^2} . \qquad (30)$$

In other words, the Green's function is finite at the coincidence limit $x = y$! As is well known, this feature can eliminate the ultraviolet divergences in quantum theory. [This is equivalent to a momentum space cut off at Planck energy]. Admittedly the arguments have to be refined further; but the physics is transparent in (23) itself.

It was always felt that Planck length should play a fundamental role in quantum gravity. Our analysis confirms this thought and shows that Planck length plays a crucial role in all physics. It provides a lower bound to all proper length scales.

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