Some New Gravitationally Conserved Quantities

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ABSTRACT

It has been commonly believed that there were no absolutely conserved quantities in the Einstein theory of gravity and that only charge was absolutely conserved in the Einstein-Maxwell theory. It is the purpose of the present note to state that this is wrong and to describe ten recently discovered absolutely conserved quantities in the Einstein theory and six conserved quantities in the Einstein-Maxwell theory. These quantities, though they have a trivial meaning in the linear theory of gravity, appear to play a fundamental role in the dynamics of the non-linear gravitational field.

SOME NEW GRAVITATIONALLY CONSERVED QUANTITIES

We shall describe a set of ten recently discovered conservation laws[1] in the Einstein theory of gravitation whose existence had been previously unsuspected. Although these conserved quantities differ fundamentally from certain counterparts in the linear theory of gravity, (which are essentially trivial) the linear theory plays an essential role here in their interpretation. We also parallel the description of the full and linear theory with an analogous description of Einstein-Maxwell theory where there are six new conserved quantities.

As a preliminary there are two mathematical points to be mentioned. First we define two sets of complete functions on the sphere (different from the spherical harmonics) as follows;

$$_1Y_{\ell m} = \sqrt{\frac{(\ell - 1)!}{(\ell + 1)!}} \left\{\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial\phi}\right\} Y_{\ell m}(\theta,\phi), \quad \ell \geqslant 1,$$

$$_2Y_{\ell m} = \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \sin\theta \left\{\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial\phi}\right\} \frac{1}{\sin\theta} \left\{\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial\phi}\right\} Y_{\ell m}, \quad \ell \geqslant 2.$$

Higher type functions, denoted $_sY_{\ell m}$, which are particularly convenient for use with spin s fields, can also be defined.

The second point concerns the formalism used. At least in the flat-space Maxwell and linearized gravitational theory one can, instead of Minkowskian coordinates, use null polar coordinates, polar coordinates θ,ϕ,r and the retarded time $u = t - r$. A generalization of this coordinate system can be introduced in the full theory when the space becomes asymptotically Minkowskian. The hypersurfaces u = const. remaining null and outgoing.

When investigating radiative solutions of Maxwell's equations it is often simpler to work with the following complex quantities than with $\vec{E}$ and $\vec{B}$;

$$\phi_o = (\vec{E} + i\vec{B})\cdot\vec{m}, \quad \phi_1 = (\vec{E} + i\vec{B})\cdot\vec{c}, \quad \phi_2 = (\vec{E} + i\vec{B})\cdot\vec{\bar{m}} .$$

The vector $\vec{m}$ and its complex conjugate $\vec{\bar{m}}$ are defined from an orthonormal triad $\vec{a}$, $\vec{b}$ and $\vec{c}$ (defined at each point of space) by $\vec{m} = \frac{1}{\sqrt{2}} (\vec{a} + i\vec{b})$. $\vec{a}$ and $\vec{b}$ are tangent vectors to the polar coordinate spheres and $\vec{c}$ is along the radial direction.

In the linear and full theory of gravitation, the quantity analogous to $F_{\mu\nu}$ is the Weyl tensor, $C_{\alpha\beta\gamma\delta}$. Just as $F_{\mu\nu}$ corresponds to $(\vec{E}, \vec{B})$, the $C_{\alpha\beta\gamma\delta}$ corresponds to two traceless symmetric three dimensional tensors. V_{ij} and W_{ij}. We define five complex quantities analogous to the above ϕ's;

$$\psi_o = (V_{ij} + iW_{ij})m_i m_j, \qquad \psi_1 = (V_{ij} + iW_{ij})c_i m_j$$

$$\psi_2 = (V_{ij} + iW_{ij})(c_i c_j + m_i \bar{m}_j), \quad \psi_3 = (V_{ij} + iW_{ij})\bar{m}_i c_j$$

$$\psi_4 = (V_{ij} + iW_{ij})\bar{m}_i \bar{m}_j \quad .$$

The differential equations satisfied by the ϕ's and ψ's can be integrated asymptotically without great difficulty. The solutions for all outgoing waves and those incoming waves which concern us have the asymptotic form;

$$\phi_o = \frac{\phi_o^\circ}{r^3} + \frac{\phi_o'}{r^4} + O(r^{-5}), \qquad \phi_1 = \frac{\phi_1^\circ}{r^2} + O(r^{-3}) \ ,$$

$$\phi_2 = \frac{\phi_2^\circ}{r} + O(r^{-2}) \quad ,$$

$$\Psi_o = \frac{\Psi_o^\circ}{r^5} + \frac{\Psi_o^1}{r^6} + O(r^{-7}), \quad \Psi_1 = \frac{\Psi_1^\circ}{r^4} + O(r^{-5})$$

$$\Psi_2 = \frac{\Psi_2^\circ}{r^3} + O(r^{-4}), \qquad \Psi_3 = \frac{\Psi_3^\circ}{r^2} + O(r^{-3}) \quad ,$$

$$\Psi_4 = \frac{\Psi_4^\circ}{r} + O(r^{-2}) \quad .$$

This general asymptotic behavior holds rigorously for the full theory and for the Einstein-Maxwell theory as well as for the linear theories.

The usual conserved quantities in the linear theories, namely charge, mass, and linear and angular momentum, can be given as surface integrals at fixed u of ϕ_1°, Ψ_2° and Ψ_1°, with appropriate ${}_sY_{\ell m}$ as weighting factors.

The new conserved quantities can be similarly defined; for the Maxwell theory they are (in complex form)

$$F_m = \int \phi_o^1 \; {}_1\overline{Y}_{1m} \, d\Omega, \qquad m = -1, 0, 1,$$

and for the linear and full theory of gravitation

$$G_m = \int \Psi_o^1 \; {}_2\overline{Y}_{2m} \, d\Omega, \quad m = -2, -1, 0, 1, 2 \quad .$$

In the full Einstein-Maxwell theory, the three F_m are still strictly conserved but the G_m (as defined above) are no longer conserved but can be "carried away" by electromagnetic radiation. If the total charge of the system vanishes, then an electromagnetic contribution R_m to the G_m can be defined such that the $G_m + R_m$ is conserved. The G_m constitute a D(2,0) representation of the homogeneous Lorentz group; they are unchanged by translations and by what is known as supertranslations.

To understand the F_m and G_m it is useful to first investigate their meaning in the linear theories. It is easy to show (in this case) that they both vanish for pure retarded fields. In addition they vanish for all advanced fields except for the dipole field in Maxwell theory and the quadrupole field in the linear gravitational case. In fact if the time profiles (of advanced time, $v = t + r$) of the advanced fields behave as $\frac{\alpha_m}{v} + O(v^{-2})$ for large v, the F_m and G_m are proportional to α_m. In other words, in the linear theories, the F_m and G_m measure the existence (in the asymptotic null future) of incoming dipole and quadrupole waves which are dying out as $1/v$.

Initially we believed that this trivial interpretation of the G_m extended to the full theory. However it was here that the non-linearity of the full theory presented an essentially new situation.

Consider the static or stationary vacuum solutions of the full theory, where incoming (or outgoing) radiation is absent according to any reasonable definition - nevertheless, the G_m do <u>not</u> vanish. They can be expressed in terms of the monopole, dipole and quadrupole moments, m, p_i and Q_{ij}, by the standard[2] relation $G_m \longleftrightarrow K_{ij}$, where

$$K_{ij} = m\, Q_{ij} - 3p_i p_j + p^2 \delta_{ij} \quad .$$

The imaginary parts of p_i and Q_{ij}, are the spin dipole and quadrupole moments respectively. K_{ij} is a trace free tensor, independent of the origin with respect to which the Q_{ij} and p_i are calculated.

This result for the stationary solutions leads to a remarkable conclusion. Suppose an initially stationary asymmetric body with $K_{ij} \neq 0$ becomes spherically symmetric at a finite time later. Since the Q_{ij} and p_i must then vanish, K_{ij} must become zero but the G_m must still correspond to the <u>original</u> values of K_{ij}. This leads to a time dependent solution containing terms very similar to the incoming waves of the linear theory, but now presumably back-scattered waves. It will not become stationary <u>in any finite time</u>. To make it stationary the system would have to develop moments such that the new K_{ij} would equal the old. We see that the G_m thus make certain classical stationary states inaccessible in a finite time from other states.

It appears therefore as if the G_m and F_m represent in addition to information about incoming waves, information about the source; the relative importance of each meaning depending on details in each case.

PHYSICAL IMPLICATIONS

We now have a set of quantities which are absolutely conserved in asymptotically flat space-times satisfying the appropriate Einstein equations and which can be measured by an examination of the asymptotic fields. The only previously known quantity of this kind was electric charge. In view of the importance to physics of the concept of charge, it seems not unreasonable to expect that there may be significant physical consequence of G_m and F_m conservation, also.

Indeed, the conservation of G_m leads immediately to one interesting conclusion concerning gravitational radiation. For the first time we have a rigorous argument which shows that gravitatonal waves emanating from an isolated system must in general undergo back-scattering. Thus a transition from a stationary state to another state with a different K_{ij} can never be achieved with clean-cut waves; the second state never achieves exact stationarity showing that there must be a residual gravitational disturbance within the light cone of the wave. (The hope had sometimes been expressed that gravitational waves might in some sense strictly obey Huygen's principle, i.e. be without tails.

When it comes to examining the role and meaning of the new conservation laws in actual physical processes, the picture unfortunately becomes more difficult and obscure. We may speculate on the role of G_m in the study of massive objects and the question of gravitational collapse, since there gravitational forces become significant in comparison with other forces. We are familiar with the idea of mass-energy conservation being of great importance in virtually any physical situation. Although conservation of mass-energy in general relativity is not so clear-cut as it is in special relativity, we can say at least the total mass-energy of an isolated system will be conserved provided we take into account a (positive) contribution to the energy which resides in the emitted gravitational waves. Now, in effect, this energy of the gravitational waves is lost to the system owing to the weakness of the gravitational coupling. During an asymmetrical collapse, large reductions in the mass of the material system may occur quite rapidly (if calculations based on the linear theory are to be believed.) Thus, mass-energy conservation will not effectively be operative at this stage. On the other hand, G_m conservation may be expected to be relatively more important since G_m is not carried away by the waves. It is possible that the behavior of a collapsing system may be severely restricted by the G_m conservation. A great deal more work needs to be done to see if the conserved quantities do lead to important physical effects in phenomenon of collapse.

An alternate possibility is that the G_m and F_m might play a role in phenomena on an extremely small scale. One might speculate as to whether the conserved quantities lead to the existence of conserved currents, (as does charge conservation) which might perhaps, act as the source of undercovered new fields. In the absence of any larger theory, such as a transformation theory which might derive the conservation laws by an application of Noether's theorem, we must at present leave the matter here.

REFERENCES

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2. J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, New York, page 99.

BIOGRAPHICAL DATA

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