The Nature of Sources of a Gravitational Field

Ezra T. Newman

Physics Department, University of Pittsburgh

Pittsburgh 13, Pennsylvania

ABSTRACT

In most field theories, it is possible to obtain solutions of the equations in a source free region of space in terms of a distant source distribution. When one attempts to do the same to Einstein's theory of gravitation great difficulties are encountered. Due to the extreme non-linearity of the equations it has so far proven impossible to determine solutions directly in terms of the sources other than by approximation techniques. We propose here a means of obtaining asymptotically exact solutions at large distances from the source and in addition a means of identifying the moments of the source.

The Nature of Sources of a Gravitational Field

Although the prime consideration in this paper will be the application of some new techniques (arising from a simple observation) to the understanding and interpretation of the gravitational field, the techniques will first be illustrated by application to the Maxwell field. It will be seen that there exists a remarkable analogy between the Maxwell field tensor and the Riemann tensor which aids in the solution of the gravitational equations and which suggests a physical interpretation for functions arising in these solutions.

The starting point of the investigation is the observation that a vector field, no matter how complicated, can appear simple by an appropriate choice of the basis system by which the vector is represented. For example, consider an orthonormal triad of vectors in three dimensions, where the first vector is always taken parallel to the arbitrary vector field $\vec{A}$. With respect to this basis $\vec{A}$ is represented by $(A,0,0)$, $A = |\vec{A}|$. A similar observation applies to tensors, e.g. a matrix represented in terms of its own eigenvectors will be particularly simple. The dimensionality of the space is arbitrary.

It is well known[1] that there are algebraically two different types of electromagnetic fields, null fields and non-null fields. The former is characterized by the vanishing of the eigenvalues of the Maxwell tensor $F_{\mu\nu}$, i.e. $E^2 - B^2 = \vec{E}\cdot\vec{B} = 0$, the latter by the non-vanishing of at least one of the eigenvalues. The non-null case is far more common and hence is the more important of the two. The non-null $F_{\mu\nu}$ possesses four eigenvectors; two are real and null, the other two are complex (linear combinations of two real space-like vectors) and also null. It is clear that with respect to this basis the $F_{\mu\nu}$ (or the $\vec{E}$ and $\vec{B}$) take a simple form.

This suggests that possibly an appropriate basis to be chosen to study the Maxwell field should consist of four linearly independent null vectors, two real ones, l^μ and n^μ, and two complex ones, m^μ and $\bar{m}^\mu$ ($\bar{m}^\mu$ is the complex conjugate of m^μ). The components of $F_{\mu\nu}$ with respect to this basis can be written as

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu$$

$$\phi_1 = \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu)$$

$$\phi_2 = F_{\mu\nu} n^\mu \bar{m}^\nu \quad ,$$

plus the complex conjugates. It is possible to write down Maxwell's equations in terms of these ϕ's[2] and the derivatives of the basis vectors. Before attempting to solve these equations, a choice must be made for the basis vectors; they can be chosen as the eigenvectors of $F_{\mu\nu}$, in which case $\phi_0 = \phi_2 = 0$, with the result that the derivatives of the basis vectors are in general very complicated; or at the other extreme all the derivatives can be made zero with the result that the ϕ's are complicated. It happens that there exists a compromise (actually more than one) which yields a very useful set of equations. The compromise consists of choosing a spatial origin with a family of light cones, the vector l^μ being tangent to the cones, and the other three vectors being propagated parallel to themselves along l^μ. (See diagram)
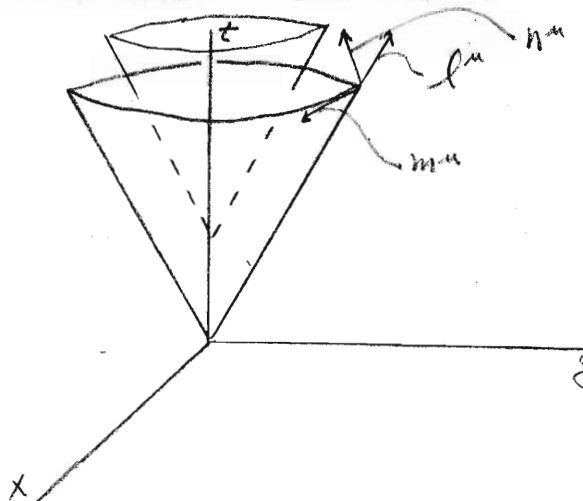
With this basis it is now possible to integrate the equations for the ϕ's. The requirement that the source of the field be bounded spatially and that we have retarded fields, leads to the result that ϕ_0 be of the order r^{-3}. (The field equations do not restrict ϕ_0 at all on one null cone.)

For simplicity we will assume

$$\phi_0 = \frac{\phi_0^{\,o}}{r^3} + 0(r^{-4}) \quad .$$

Upon integration of half the field equations, we obtain

$$\phi_1 = \frac{\phi_1^{\,o}}{r^2} + 0(r^{-3})$$

$$\phi_2 = \frac{\phi_2^{\,o}}{r} + 0(r^{-2}) \quad .$$

The order of magnitude symbols used in ϕ_1 and ϕ_2 can if needed be expressed completely in terms of ϕ_0. $\phi_1^{\,o}$ and $\phi_2^{\,o}$ (functions of θ, ϕ and the t at the apex of the cone) are "constants" of integration. The second half of the field equations determines the dependence of ϕ_1 and ϕ_0 on the retarded time t in terms of $\phi_2^{\,o}$. $\phi_2^{\,o}$ is given as an arbitrary function of θ, ϕ and t. To summarize the results of the integration, we have the following; ϕ_0 is chosen arbitrarily on one null cone subject only to being $0(r^{-3})$, and the r dependence of ϕ_1 and ϕ_2 follows by explicit integration. The leading term in ϕ_2, i.e. $\phi_2^{\,o}$, can be chosen as an arbitrary function of θ, ϕ and t. The t dependence of ϕ_1 and ϕ_0 follows from that of $\phi_2^{\,o}$. This constitutes a solution of the null or characteristic initial value problem for Maxwell theory. It should be noted that physically the arbitrariness in $\phi_2^{\,o}$ corresponds to the arbitrariness of the news or information sent out by a broadcasting station.

In order to have a better understanding of this formalism and the physical meaning of the ϕ's, it is useful to express well known solutions in terms of the ϕ's. The Coulomb or monopole field is thus

$$\phi_0 = 0$$

$$\phi_1 \alpha \frac{e}{r^2}$$

$$\phi_2 = 0 \ .$$

The static dipole field is given by

$$\phi_0 \ \alpha \ \frac{p \sin\theta}{r^3}$$

$$\phi_1 \ \alpha \ \frac{p \cos\theta}{r^3}$$

$$\phi_2 \ \alpha \ \frac{p \sin\theta}{r^3} \ ,$$

where a real and imaginary p are, respectively, the electric and magnetic dipole moments. The time-dependent dipole field is

$$\phi_0 \ \alpha \ \frac{p \sin\theta}{r^3}$$

$$\phi_1 \ \alpha \ \frac{\dot{p} \cos\theta}{r^2} + O(r^{-3})$$

$$\phi_2 \ \alpha \ \frac{\ddot{p} \sin\theta}{r} + O(r^{-2})$$

with the same meaning as above for p. Note that the leading term in ϕ_2 gives the well known result that the radiation field is proportional to the second time derivative of the dipole moment. Similar results hold for arbitrary multipole fields.

In the study of the gravitational field, the Riemann tensor is the quantity which will be taken as analogous to the Maxwell field tensor $F_{\mu\nu}$. In general there are twenty independent components of the Riemann tensor $R_{\alpha\beta\gamma\delta}$. If, however, we restrict ourselves to the case where the Einstein empty-space equations $R_{\mu\nu} = 0$ hold, we are left with ten independent components. We can define five complex components of the Riemann tensor (analogous to the ϕ's) with respect to the same type of basis that was

used in the Maxwell case, as follows:

$$\Psi_0 = R_{\alpha\beta\gamma\delta} \, l^\alpha m^\beta l^\gamma m^\delta$$

$$\Psi_1 = R_{\alpha\beta\gamma\delta} \, l^\alpha m^\beta l^\gamma n^\delta$$

$$\Psi_2 = R_{\alpha\beta\gamma\delta} \, l^\alpha m^\beta \bar{m}^\gamma n^\delta$$

$$\Psi_3 = R_{\alpha\beta\gamma\delta} \, l^\alpha n^\beta \bar{m}^\gamma n^\delta$$

$$\Psi_4 = R_{\alpha\beta\gamma\delta} \, n^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta \quad .$$

These five Ψ's satisfy eight differential equations[2] (the so-called Bianchi Identities) which are very similar in form to the equations for the ϕ's. It is possible to make a choice of tetrad and coordinates r, θ, ϕ and a retarded time t, again similar to that used in the Maxwell case, that allows one to solve the equations in the asymptotic region of large r.[2],[3] For bounded sources Ψ_0 must be taken as $O(r^{-5})$. The first four equations can then be integrated with the result

$$\Psi_0 = \frac{\Psi_0^0}{r^5} + O(r^{-6})$$

$$\Psi_1 = \frac{\Psi_1^0}{r^4} + O(r^{-5})$$

$$\Psi_2 = \frac{\Psi_2^0}{r^3} + O(r^{-4})$$

$$\Psi_3 = \frac{\Psi_3^0}{r^2} + O(r^{-3})$$

$$\Psi_4 = \frac{\Psi_4^0}{r} + O(r^{-2}) \quad .$$

The second four equations yield the time dependence of Ψ_0, Ψ_1, Ψ_2 and Ψ_3, all in terms of Ψ_4^0. The initial value data is then seen to

be Ψ_0 as a function of r, θ and ϕ on one cone and $\Psi_4{}^0$ as a function of θ, ϕ and t. This constitutes a solution of the null or characteristic initial value problem for the Einstein theory of gravitation. The meaning one gives to the arbitrariness of $\Psi_4{}^0$, is that it constitutes the "news" from the gravitational broadcasting station.

The multipole moments in electrodynamics are defined in terms of the charge-current distribution. For a given charge-current distribution the Maxwell equations can be integrated and the solutions stated in terms of the associated moments. The analogous procedure can not be followed in the theory of relativity due to the extreme non-linearity of the field equations and the consequent difficulties of integrating from the source to infinity. Normally the reverse procedure is followed; a solution of the field equations is found, then a guess is made about the nature of the source. (The guess is frequently a good one, based on studies of the symmetries and other properties of the solution.) It is clear that a definite prescription for deciding on the nature of the source of the gravitational field would be of great value. We here propose one for the case of stationary fields. (The general prescription for arbitrary time dependent fields, which will be given elsewhere,[4] is not very different.)

The monopole moment or mass is given by the real part of $\Psi_2{}^0$. The mass dipole moment and the angular momentum are proportional to the real and imaginary parts respectively, of $\Psi_1{}^0$. The quadrupole and higher moments are obtained from the coefficient of the r^{-5}, r^{-6}, etc. terms in Ψ_0.

The reason the multipole moments are defined in this fashion is threefold. If we integrate the linearized Einstein equations with sources, the moments, defined from the sources, enter into the solutions as in our definition. The second reason is given simply through the

the remarkable analogy between the Maxwell and Einstein equations, the type of initial data to be chosen in both theories, the similarity of the asymptotic behavior of the ϕ's and Ψ's and (not shown in this paper) the similarity between the transformation properties of the ϕ's and the Ψ's. It is therefore reasonable to suppose that the moments can be defined similarly. The last reason is that this definition conforms with that obtained from the study of the symmetry properties of special solutions. For example, Ψ_2 is the mass in the Schwarzschild or monopole (spherically symmetric) solution, the other moments being zero. A study of the symmetries and singularities of Kerr's spinning particle solution allows us to consistently interpret the imaginary part of Ψ_1^0 as the angular momentum and the real part of Ψ_0^0 as the mass quadrupole of the source. [5]

At present an analysis of many other solutions, including the Weyl-Levi-Civita solutions, is being carried out using these definitions.

Acknowledgment

References

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Biographical Sketch

of

Ezra T. Newman

BORN — October 1929 in N.Y.C.

DEGREES — B.A. 1951, N.Y.U.

Ph.D. 1956, Syracuse University

THESIS SUBJECT - General Theory of Relativity

EMPLOYMENT — Instructor, Assistant Professor, Associate
Professor, University of Pittsburgh

LEAVE OF ABSENCE — Syracuse University 1960-61

University of London, Kings College 1964-65

MAJOR SUBJECT OF RESEARCH - Theory of Gravitational Radiation