Soliton concept in General Relativity

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Summary

Soliton physics has made considerable progress in solving non-linear problems. This paper is meant to relate the soliton concept to the stationary axisymmetric vacuum fields in General Relativity. We present a functional transformation which, working as a non-linear creation operator, generates gravitational fields of isolated sources. When applied to flat space-time ('gravitational vacuum') this operation leads to a non-linear superposition of an arbitrary number of Kerr particles.

This superposition also includes the Tomimatsu-Sato fields.

The functional transformations form an infinite-parameter group which contains the Kinnersley-Geroch group as a subgroup.

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1. Introduction

Soliton physics has provided us with valuable knowledge in the
field of non-linear phenomena. We have learned that there are
stable structures formed and maintained by a competition between
dispersion and non-linear feedback. Such soliton structures are
being investigated in many areas of physics, e.g. in non-linear
optics, plasma physics, and dislocation theory.
It is the purpose of this essay to relate the soliton concept
to Einsteinian gravity and to report the results achieved in
this way.
Let us consider the stationary axisymmetric vacuum gravitational
field of a spinning mass distribution.
In General Relativity, this problem can be described by the
complex Ernst potential f , which is an <u>axisymmetric</u> solution
of the second-order differential equation

$$(Re \ f) \ \triangle f \ = (grad \ f \ )^2 \quad , \tag{1}$$

where $\triangle$ is the Laplacian operator in Euclidean 3-space.
Before we discuss the axisymmetric class of solutions let us
have a look at two other subclasses of eq. (1). One of them shall
us/show the way on which we can find the wanted axisymmetric
soliton solutions, the other one shall inspire the physical inter-
pretation.

(i) In the case of <u>cylindrical</u> symmetry, in which f depends on two Cartesian coordinates, eq. (1) is equivalent to two elliptic sine-Gordon equations. The usually considered hyperbolic sine-Gordon equation has multi-soliton solutions, which can be constructed by means of successive Bäcklund transformations. The point is that the Bäcklund transformation method, which also applies to the elliptic type, essentially involves algebraic manipulations. These facts have stimulated the search for a simillar procedure applicable to the axisymmetric solutions of eq. (1). The search proved successful /1/.

(ii) Newtonian gravitational fields are characterized by Im f=0 , in which case eq. (1) reduces to the Laplace equation

$$\triangle \phi = 0 \qquad (2)$$

for the real gravitational potential $\phi = 1/2 \ln f$. Newtonian theory is very suitable for illustrating the effect of Bäcklund transformations on gravitational fields, and the study of Newtonian solitons (sec.2) will give us valuable hints how to proceed in Einstein's theory (sec.3).

2. Solitons in Newton's theory

Let $\phi(z,\bar{z})$ be a known axisymmetric gravitational potential depending on the cylindrical coordinates $\rho = \mathrm{Re} z$, $\zeta = \mathrm{Im} z$, where ρ measures the distance from the symmetry axis (ζ - axis). Then the operation $I_K^{\pm}$,

$$I_K^{\pm} : \quad \phi'(z,\bar{z},K) = \pm \int (\lambda \, \phi_{,z} \, dz + \lambda^{-1} \, \phi_{,\bar{z}} \, d\bar{z}) ,$$

$$z' = iKz(z - iK)^{-1} , \qquad (3)$$

where

$$\lambda = \lambda(K) = (K - i\bar{z})^{1/2} \cdot (K + iz)^{-1/2} , \qquad (4)$$

transforms the original potential ϕ into a function ϕ' of $z, \bar{z}$ and K. (A bar denotes complex conjugation). Obviously, the operation $I_K^{\pm}$ consists of a path-independent integral transformation and a coordinate transformation. Replacing the arguments $(z, \bar{z})$ of ϕ' by the new coordinates $(z', \bar{z}')$ and reinterpreting $\text{Re} z'$ and $\text{Im} z'$ as cylindrical coordinates, we obtain a new axisymmetric solution ϕ' of the potential equation (2). A second symmetry operation S,

$$S : \qquad \phi' = -\phi + \frac{1}{2} \ln(\text{Re} z), \qquad (5)$$

changes the sign of the original mass density (first term) and superpones an infinite rod which covers the symmetry axis with the line density $\mu = 1/2$ (second term).

The composed operation $B_K^{\pm} = SI_K^{\pm}$ is usually called a Bäcklund transformation (BT) with the parameter K. Let us discuss the effect of successive BT's. The particular double Bäcklund transformation $B_{K_1}^- B_{K_1}^+$ invertes the sign of the original potential ϕ and superpones the potential of a rod covering the infinite interval $(\zeta = K_1, \zeta = +\infty)$ on the symmetry axis. Using the terminology of the BT method we denote the gravitational field of the rod as a soliton.

The further double Bäcklund transformation $B_{K_2}^- B_{K_2}^+$, with $K_2 < K_1$,

(i) restores the sign of the original potential and changes the sign of the rod mass density ($\mu = 1/2 \to \mu = -1/2$) and

(ii) superpones a second rod which covers the interval $(K_2, +\infty)$ with a positive mass density $\mu = 1/2$.

In the common region of the rods, the positive and negative masses compensate, and we have a finite rod with the length $(K_1 - K_2)$, the mass density $\mu = 1/2$, and the mass $m = (K_1-K_2)/2$.

A successive application of the four-fold $\sqrt{}$ ■ generates the superposition of the original solution and an arbitrary number of rod potentials.

Backlund transformation $B_{K_2}^- B_{K_2}^+ B_{K_1}^- B_{K_1}^+$

3. Solitons in Einstein's theory

This result may be translated into Einstein's theory, in which the static axisymmetric solutions are also described by the Laplace equation (2). In Weyl's cylindrical coordinates, the mass rod potential represents the famous Schwarzschild solution, which now turns out to be a double soliton. Therefore, $B_{K_2}^- B_{K_2}^+ B_{K_1}^- B_{K_1}^+$ is a creation operator for Schwarzschild particles (black holes). Involving limiting procedures one can construct any axisymmetric static asymptotically flat Einstein field by a superposition of Schwarzschild masses distributed along the symmetry axis. The point made here is that the symmetry operations $I_K^{\pm}$ and S are sufficient to generate the exterior field of any isolated source which has the said symmetries.

Now let us return to the axisymmetric stationary fields and try to find a generalization of $I_K^{\pm}$ and S. Indeed, such a generalization does exist /1/. Following Refs. /1-2/ we could repeat the procedure outlined for Newtonian fields step by step. Here we shall be concerned solely with a discussion of the result. After N four-fold BT's one obtains from a given Ernst potential f the following new Ernst potential f' /3/:

$$f' = f \ \frac{D \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{2N} \\ 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2N} \end{pmatrix}}{D \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{2N} \\ 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2N} \end{pmatrix}} \quad , \tag{6}$$

where

$$D \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{pmatrix} = \det \left\{ \exp \left[\left(\frac{1+(-1)^r}{2} \right) \ln x_s + (r-1)\ln y_s \right] \right\} \tag{7}$$

$$(r, s = 1 \cdots m)$$

is a Vandermonde-like $(m \times m)$ determinant. The symbol $\lambda_s = \lambda(K_s)$ was already explained in eq. (4). We have to choose $\alpha_0 = -\bar{f}/f$. The quantity α_s (s = 1,2 ... 2N) is a functional of the original solution f and can be calculated from f via an ordinary Riccati differential equation /1,3/. α_s depends on the coordinates $(z,\bar{z})$, the constant K_s entering λ_s, and an additional integration constant L_s: $\alpha_s = \alpha(z,\bar{z}; K_s, L_s)$. For instance, if the original solution is static $(f = e^{2\phi} = \bar{f})$, the functional α_s is given by

$$\alpha_s = \frac{L_s + i \ e^{2\phi'_s}}{L_s - i \ e^{2\phi'_s}} \tag{8}$$

with $\phi'_s = \phi'(z,\bar{z};K_s)$ as in eq. (3). If f corresponds to an asymptotically flat space-time, the constants L_s, K_s can be chosen in such a way that the space-time connected with f is also asymptotically flat. In the sense of the preceding considerations the potential f' describes a non-linear superposition of the original potential f and N "spinning rod" potentials which turn out to be Kerr-NUT solutions. It should be noted that an angular momentum monopole (an NUT singularity) can be removed by means of an Ehlers transformation. The pure 2N-soliton solution which is a non-linear superposition of N Kerr solutions originates from the Minkowski vacuum $(\phi = 0)$. Using eqs.(3) and (8) it can readily be verified that, for $\phi = 0$, the functions α_s

are constants. There are two different cases to be distinguished.
The constants K_s in λ_s must be either purely real or arise
as complex conjugated pairs $\overline{K_i} = K_j$ ($i \neq j$). The first case is
well known from Newton's theory, cf. eq. (4), and implies
$\overline{\alpha}_s = \alpha_s^{-1}$, $\overline{\lambda}_s = \lambda_s^{-1}$. The other one leads to the hyper-
extreme Kerr solutions which have no Newtonian or static counter-
part. In order to get the superposition of N Kerr black holes
(real K_s) we set

$$\alpha_0 = -1, \quad \alpha_s = (-1)^{s+1} exp\left[i(-1)^s \varphi_s\right], \quad \varphi_{2s} = \varphi_{2s-1} \quad (s = 1, \ldots, N). \quad (9)$$

The real constants φ_{2s} are the rotation parameters for the
individual Kerr particles. The masses of these particles are
given by

$$m_s = \frac{1}{2}(K_{2s} - K_{2s-1}) \, cos\varphi_{2s-1} \, . \quad (10)$$

It is an exciting task to find out whether the gravitational
attraction of the Kerr black holes can be balanced by a rotational
repulsion effect, cf. /4/. In order to obtain the superposition
of N Schwarzschild particles we put $\varphi_s = 0$. In this case,
numerator and denominator on the right-hand side of eq. (6) become
Vandermonde determinants and can therefore be factorized. This
solution exactly agrees with the result of N fourfold Newtonian
BT's applied to $\phi = 0$. If the original solution f belongs to
the static Weyl class, eq. (8) applies. This solution corresponds
to an asymptotically flat space-time involving the harmonic func-
tion ϕ, and N mass and angular momentum parameters. In the
limit $N \to \infty$, this solution might be a suitable starting point
for solving any boundary condition problem for isolated sources.
For increasing, but finite N it approximates the solution of

such a problem by a set of <u>exact</u> axisymmetric solutions.

At present the 2N-soliton transformation (6) provides us with the most comprehensive explicitly given Ernst potential f' .

Let us consider some of its special cases:

(i) The Tomimatsu-Sato $\delta = N$ solution is the superposition of N identical Kerr solutions with coinciding sources,

$$K_1 = K_3 = K_5 = \ldots = K_{2N-1}; \quad \varphi_1 = \varphi_3 = \ldots = \varphi_{2N-1} ;$$

$$K_2 = K_4 = K_6 = \ldots = K_{2N} ; \quad \varphi_2 = \varphi_4 = \ldots = \varphi_{2N} .$$

In this case, N pairs of columns of the determinants in eq. (6) coincide so that the numerator and the denominator of the fraction become zero. So the Bernoulli-l'Hospital rule applies (N-1) times, cf. /4/. No wonder that the explicitly written result is rather complicated.

(ii) The functional transformations (6), including their limits $N \to \infty$, form an infinite-parameter group F , which contains the Kinnersley-Geroch group K as a subgroup. To obtain representations of this subgroup, again we have to deal with coincidences of determinant columns. For instance, we may generate the 'exponentiated' Hoenselaers-Kinnersley-Xanthopoulos (HKX-)transformation /5/ by setting $K_1 = K_2 = \ldots = K_{2N}$, $L_1 = L_2 = \ldots = L_{2N}$, and applying the Bernoulli-l'Hospital rule N times. In the lowest order (N = 1, HKX rank zero), the HKX transformation generates, from the vacuum (f = 1) , the extreme Kerr solution, whereas our group operation leads to the general Kerr metric.

As it was shown by Cosgrove /6/, our present method is equivalent to the inverse scattering approach of Belinsky and Zakharov /7/.

These authors already calculated the Kerr solution from flat space-time. Their formalism implicitly contains the 2N-soliton solution, but they did not carry the analysis far enough to reach an explicit expression like eq. (6).

Finally, we mention that not only the Ernst potential (6) but also the full metric can be given explicitly in terms of the original metric and quantities associated with it. The derivation of the corresponding formulae, which mainly consist of determinants (7), requires only algebraic operations. The details will be published elsewhere.

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BIOGRAPHICAL SKETCH

Dietrich K r a m e r

I was born in Weida (GDR) in 1939 and attended the primary and
secondary schools in my birth-place. From 1957 to 1962 I studied
physics at the Friedrich Schiller University of Jena (GDR), where
I took doctor's degree in 1966 and qualified for a lectureship
("Habilitation") in 1970. Since that time I lecture on Theoreti-
cal Physics in Jena. As a member of Professor Schmutzer's relati-
vity group I was especially concerned with spinor fields and
quantization in curved space-time and with exact solutions of
Einstein's field equations. For some years, the latter topic was
the common research subject of Dr. G. Neugebauer and myself;
at present we are working again on that matter. Together with
Drs. H. Stephani and E. Herlt from the University of Jena, and
Dr. M.A.H. Mac Callum from the Queen Mary College of London,
I have written a forthcoming book on exact solutions.

BIOGRAPHICAL SKETCH

Gernot N e u g e b a u e r

G. Neugebauer was born in 1940. After he had finished the
secondary school in 1958 he studied physics at the Friedrich
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till 1963, since when he has been working there. In Pro-
fessor Schmutzer's group he dealt with questions of relati-
vistic thermodynamics and other relativistic problems, e.g.
exact solutions of Einstein-Maxwell equations, the latter
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1966 and qualified for a lectureship ("Habilitation") in
1970. Recently he has worked in the field of statistical
thermodynamics and soliton physics.