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Summary

We present a new *finite* action for Einstein gravity in which the Lagrangian is quadratic in the covariant derivative of a spinor field. Via a new spinor-curvature identity, it is related to the standard Einstein-Hilbert Lagrangian by a total differential term. The corresponding Hamiltonian, like the one associated with the Witten positive energy proof is fully 4-covariant. It defines quasi-local energy-momentum and can be reduced to the one in our recent positive energy proof.

A Quadratic Spinor Lagrangian for General Relativity

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The gravitational field responds to and exchanges energy-momentum with its sources. The matter and field sources are described by a proper energy-momentum density tensor. The energy-momentum density of the gravitational field itself, however, has proved to be elusive. To find a good mathematical expression for the gravitational energy-momentum density has been a major impetus in the development of new techniques for representing Einstein's gravity theory.

The prime principle of theoretical physics is to begin with the action. For Einstein's General Relativity the usual starting point is the Hilbert action in which the Lagrangian is the scalar curvature. But the scalar curvature contains second derivatives of the metric with an associated asymptotically flat space fall off of $O(1/r^3)$ which causes the action to diverge. One can improve the situation while removing the second derivatives of the metric by extracting a total differential. However the remaining action is no longer covariant; consequently, the energy momentum density constructed from it is a non-covariant object — a pseudotensor — which can have any local value (including zero or *negative* values). After much effort along these lines many have given up on the idea of constructing a reasonable gravitational energy-momentum density by starting from the Lagrangian or in some other way. Yet others find that their physical intuition expects at least a quasi-local gravitational energy-momentum density. Ideally it should be connected to the action.

Here we present a finite action for Einstein gravity with a 4-dimensionally covariant Lagrangian which leads to a 4-covariant Hamiltonian which, in turn, permits not only a positive energy proof but also a locally positive energy density as well as values for quasi-local quantities.

We achieve this by introducing an auxiliary spinor field. The key is a newly discovered spinor-curvature identity [1] which permits the aforementioned extraction of a total differential to be done in a covariant fashion. In gravity research the efficacy of spinor techniques (see e.g., [2,3,4]) has long been recognized and they have been used extensively for both utilitarian reasons as well as for affording insights. Nevertheless, in this widely explored topic one can still discover new things.

In this work, we begin with a quadratic-spinor action which yields the Einstein field equations. Then we obtain a covariant expression for the Hamiltonian. This serves as our energy-momentum density providing a positivity proof and localization that has links with the Witten proof as well as others.

The new *quadratic spinor* action is given by:

$$S[\Psi] = \int \mathcal{L} = \int 2D\overline{\Psi}\gamma_5 D\Psi \tag{1}$$

where the gravitational variable is a spinor-valued 1-form field $\Psi = \vartheta\psi$, which includes an orthonormal frame 1-form $\vartheta := \gamma_\alpha \vartheta^\alpha$ and a normalized spinor field ψ (i.e. $\overline{\psi}\psi = 1$). The covariant derivative $D\Psi := d\Psi + \omega\Psi$ includes the matrix† valued connection one-form $\omega := \frac{1}{4}\gamma_{\alpha\beta}\omega^{\alpha\beta}$. (Clifford algebra valued forms [5] permit very succinct representations.)

The new *spinor-curvature identity* [1],

$$2D\overline{\Psi}\gamma_5 D\Psi \equiv -\overline{\psi}\psi\, R *1 + d[(D\overline{\Psi})\gamma_5\Psi + \overline{\Psi}\gamma_5(D\Psi)] \tag{2}$$

reveals that the quadratic-spinor Lagrangian differs from the standard Einstein-Hilbert Lagrangian only by a total differential term. Hence they yield the same field equations. However, the new quadratic spinor Lagrangian is asymptotically $O(1/r^4)$ which guarantees finite action, an advantage over the Einstein-Hilbert $O(1/r^3)$.

From the new Lagrangian, by variation with respect to $\overline{\Psi}$, we obtain Dimakis and Müller-Hoissen's [5] "Clifform" transcription of the (vacuum) Einstein field equations:

$$\frac{\delta\mathcal{L}}{\delta\overline{\Psi}} = -2\gamma_5 D^2\Psi = -2\gamma_5\Omega\Psi = -\tfrac{1}{2}\Omega^{\alpha\beta}\wedge\vartheta^\mu\gamma_5\gamma_{\alpha\beta}\gamma_\mu\psi = G_{\alpha\beta} *\vartheta^\alpha\gamma^\beta\psi = 0, \tag{3}$$

where $\Omega := d\omega + \omega\wedge\omega = \frac{1}{4}\gamma_{\alpha\beta}\Omega^{\alpha\beta}$ is the matrix valued curvature 2-form.

The Hamiltonian can be constructed [6,7] from the action by choosing a timelike evolution vector field N such that $i_N dt = 1$ and splitting the action: $S = \int\mathcal{L} = \int dt \int i_N\mathcal{L}$.

† The Dirac matrix conventions are $\gamma_{(\alpha}\gamma_{\beta)} = g_{\alpha\beta}$, $\gamma_{\alpha\beta} := \gamma_{[\alpha}\gamma_{\beta]}$, $\gamma_5 := \gamma^0\gamma^1\gamma^2\gamma^3$. The metric signature is $(+---)$.

This procedure yields the Noether translation generator along N, i.e., the 4-covariant Hamiltonian 3-form:

$$\mathcal{H}(N) = 2[D(\overline{\psi}\, \not{N})\gamma_5 D(\vartheta\psi) + D(\overline{\psi}\vartheta)\gamma_5 D(\not{N}\psi)]. \qquad (4)$$

A notable feature is that the Hamiltonian (4) is already $O(1/r^4)$. Consequently, its integral will be finite. Moreover its variation will have an $O(1/r^3)$ boundary term which will vanish asymptotically — there is no need for a further adjustment by an additional boundary term [8]. Indeed, although it is not readily apparent, the Hamiltonian expression (4) could be obtained from the usual (linear in the Einstein tensor) Hamiltonian by adding a certain total differential (although important for the value of energy-momentum such a total differential does not effect the equations of motion) as the following identity reveals:

$$\mathcal{H}(N) \equiv -2\overline{\psi}\psi N^\mu G_{\mu\nu} *\vartheta^\nu + 2d[\overline{\psi}\, \not{N}\gamma_5 D(\vartheta\psi) + D(\overline{\psi}\vartheta)\gamma_5\, \not{N}\psi]. \qquad (5)$$

This identity also shows that the dervatives of ψ itself are not so important — up to an exact differential $\mathcal{H}(N)$ is algebraic in ψ — rather that these factors arrange for the correct quadratic connection terms.

The Hamiltonian 3-form $\mathcal{H}(N)$ is similar to the one [6]

$$\begin{aligned}\mathcal{H}_w &= 2[D\overline{\psi}\gamma_5 D(\vartheta\psi) + D(\overline{\psi}\vartheta)\gamma_5 D\psi] \\ &\equiv -2N^\mu G_{\mu\nu} *\vartheta^\nu + 2d[\overline{\psi}\vartheta\gamma_5 D\psi - D\overline{\psi}\gamma_5\vartheta\psi] \qquad (6)\end{aligned}$$

associated with the Witten positive energy proof [3]. The principal procedural difference is that the latter was obtained from the usual Hamiltonian by reparameterization and discarding a boundary term; unlike $\mathcal{H}(N)$ (4), it cannot be obtained from a 4-covariant Lagrangian. The reason for this lies in the principal technical difference: in eq (4) N^μ and ψ are independent, whereas in eq (6) the time evolution vector field, $N^\mu = \overline{\psi}\gamma^\mu\psi$, is tied to the spinor field.

The amount of energy-momentum within a region can be determined from the value of the Hamiltonian. With Einstein's field equations satisfied, the Hamiltonian 3-form (5) reduces to an exact differential which becomes an integral over the 2-surface bounding the region:

$$E = H(N)_{\rm sol} = \int \mathcal{H}(N)_{\rm sol} = \oint 2N^\alpha (\overline{\psi}\gamma_5\gamma_{\alpha\beta}\vartheta^\beta \wedge D\psi - D\overline{\psi} \wedge \gamma_5\vartheta^\beta\gamma_{\beta\alpha}\psi). \tag{7}$$

This expression is comparable to the corresponding Witten energy expression:

$$E = -2 \oint (\overline{\psi}\gamma_5\vartheta \wedge D\psi + D\overline{\psi} \wedge \gamma_5\vartheta\psi). \tag{8}$$

A variety of expressions of this type have been used in quasi-local energy investigations [9,10]. For any boundary values our new 4-covariant version (7) likewise yields a quasi-local energy. We have, as yet, no new ideas concerning the important question of how to select the best values on the boundary. In addition to being 4-covariant the new expression has the merit of being connected with a Lagrangian.

Our new quadratic-spinor formulation permits a simple locally positive expression for the Hamiltonian density. In eq (4) we choose ψ to conformally satisfy the Witten equation, $\gamma^a D_a(f\psi) = 0$, the shift vector to vanish and the lapse $N = f^2$. A further algebraic restriction on ψ then reduces the Hamiltonian density to

$$\mathcal{H} = f^2[4g^{ab}\nabla_a\varphi^\dagger\nabla_b\varphi + K^{ab}K_{ab} - K^2], \tag{9}$$

which is the same simple locally positive (on maximal surfaces) expression found in our recent positive energy proof. It has links to both the *special orthonormal frame* (SOF) and the Witten proofs and their associated energy localizations [11].

In summary we have essentially presented a covariant version of the procedure of removing a total derivative from the Einstein-Hilbert action: $R \sim \partial\Gamma + \Gamma\Gamma$. We have used a spinor field to achieve this covariantization. The quadratic spinor action considered here has an $O(1/r^4)$ fall off which makes the action converge. From it we obtained a fully

4-covariant Hamiltonian which has an associated well-defined boundary term that gives both total conserved and quasi-local quantities for gravitating systems. The variables can be selected so that the Hamiltonian reduces to the one used in our recent positive energy proof which has links to both the SOF and Witten approaches.

Perhaps it is worth remarking that we have not given any physical interpretation to our spinor field. The spinor field in our quadratic spinor action is merely used here to provide a way of *covariantization* of the $O(1/r^4)$ Lagrangian and Hamiltonian.

Of course there are variations on our theme. One can easily generalize to the Einstein-Cartan theory by allowing non-vanishing torsion. It is also possible to consider instead the (anti) self-dual connection and curvature. This would connect with work on the New Variables. Moreover, our new form of the action for Einstein's theory may yield other benefits in addition to those considered here concerning gravitational energy and its localization.

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