GRAVITATIONAL POTENTIALS: A CONSTRUCTIVE APPROACH TO GENERAL RELATIVITY

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Summary

Recent investigations of the initial-value problem of general relativity have shown that the initial-value constraints can be formulated in all cases as a system of elliptic equations with well defined physical and mathematical properties. The solutions of these equations can be regarded as generalized gravitational potentials. These potentials are interrelated and depend on their sources quasi-linearly. They are particularly useful in analyzing asymptotically flat solutions of Einstein's equations. We have found from these results (1) a technique for constructing physically meaningful initial data in the integration of Einstein's equations, and (2) a method for characterization and analysis of the spacelike mass, momentum, angular momemtum, and multipole moments of gravitational fields.

If one wishes to analyze Einstein's equations as a dynamical system, one must introduce a notion of "time." This is achieved, without the introduction of special coordinate systems, by foliating the spacetime manifold with a family of spacelike hypersurfaces.[1] The foliation may be arbitrary, except for its spacelike character. The importance of these slices rests on the fact that complete Cauchy data for the gravitational field can be defined on any one slice. Given, in any coordinate system, the intrinsic geometry of the slice in terms of a metric g_{ij}, and its extrinsic geometry in terms of the second fundamental tensor K_{ij}, one has necessary and sufficient information to construct a unique solution of Einstein's equations. Of course, one must complement these data with initial data characterizing matter or field sources, in order to generate non-vacuum spacetimes.

The difficulty, and the interest, in the gravitational initial-value problem arises from the fact that the initial data cannot be given arbitrarily, but must be chosen to satisfy four constraints. Our recent studies[2] of the problem have shown that the constraints can be reduced to a system of four quasi-linear elliptic equations when one makes a suitable choice of the independent and dependent parts of the initial data. The ellipticity of the equations is important on two counts. Firstly, it means that the equations have strong existence and uniqueness properties[3] because many of the well-developed techniques and theorems concerning elliptic systems can be applied. For a general choice of independent data the equations can be solved to find the dependent parts and hence a complete compatible set of Cauchy data. It is of particular interest that the equations are in a form that can be readily adapted to numerical solution on a computer.[4] This is, of course, of great importance owing to the analytical intractibility of Einstein's equations in the great majority of cases. Thus, one can use this program of solving four of the ten Einstein equations constructively to begin to "build" a spacetime with well-defined physical properties, rather than just to analyze a pre-existing complete spacetime solution. Such complete spacetimes are simply not available in general. Moreover, one is not limited with this method to perturbations around known solutions.

The mathematical properties of the initial value problem help one to characterize the physical structure of the gravitational field also.[5] This is the second aspect of the importance of the ellipticity of the equations that we wish to discuss in this essay. Since the equations are elliptic, the dependent data can be regarded as gravitational potentials. These potentials are generalizations of the standard potential functions that arise in Newtonian theory. The concept of generalized potentials is particularly fruitful when the spacetime is asymptotically flat, and we will limit our remarks to this case for the remainder of this essay.

Since quantities such as energy, momentum and angular momentum are constants of the motion, one expects that they can be expressed naturally in terms of the initial data. Expressions for these quantities were found by Arnowitt, Deser and Misner[6] and by Brill.[7] They showed that each of these objects can be expressed by surface integrals of part of the initial data at spatial infinity. This alone is reason to expect the existence of suitable potentials because we know that the total energy, momentum, and angular momentum should depend directly on the interior distribution of all the initial data for the gravitational field and its sources.

Well-chosen independent data can be assumed in many cases to fall off sufficiently rapidly that the asymptotic behavior of the field can be expressed solely in terms of the potentials. Our <u>closed-form</u> elliptic equations, together with a new and well-defined specification of "almost symmetry vectors" in the space-like slice, now supply a rigorous connecting link between the interior and asymptotic data by means of exact, three-dimensionally coordinate-free "Gaussian" theorems.[2] There is no need to rely on exact isometries of the space-like slice, nor to postulate timelike symmetry vectors throughout the spacetime. Indeed, the latter cannot be known a priori in the constructive approach. In particular, what we are doing is making an expansion of the generalized potentials in terms of the multipoles of the elliptic operators. The energy, momentum and angular momentum discussed above are particular terms in the multipole expansion, and can be analyzed in detail without depending on exact isometries of spacetimes, which would limit one to static and stationary spacetimes, and their perturbations.

This entire analysis is based on choosing the generalized potentials as a scalar ϕ, which scales the three-metric, and a vector W^i, which generates the longitudinal part of the extrinsic curvature K_{ij} through the conformal Killing form $(LW)^{ij} = \nabla^i W^j + \nabla^j W^i - \frac{2}{3} g^{ij} \nabla_k W^k$. The independent data now is the conformal three-geometry, the transverse-tracefree part of K_{ij} and the trace of K_{ij}. The initial value constraints take the form[8]

$$8 \nabla^2 \phi + (LW)^2 \phi^5 = [\text{gravitational terms}] - 16\pi \mu \phi^5$$

$$\nabla_j (LW)^{ij} + 6 (LW)^{ij} \nabla_j \ln \phi = [\text{gravitational terms}] - 8\pi v^i \phi^4$$

Here, μ is the source energy density and v^i is its current density. "$\nabla . L$" is a covariant vector Laplacian that is both strongly elliptic and self-adjoint.[9] Hence ϕ is a mass potential and W^i is a current potential.

Multipole moments are well-defined to the extent the right-hand sides of the above equations vanish quickly enough as $r \to \infty$.[10] This places restrictions both on the ordinary sources and on the gravitational waves ("getting beyond the wavefront"). Let us assume that the independent data has compact support. Hence, in the far-field the right-hand sides vanish. The resulting equations are still both non-linear and coupled. However, if ϕ and W^i are expanded in two series in powers of $(1/r)$ and we write out the equations term-by-term, equating powers of (1/r), the result is a series of equations, each of which is linear and appropriately decoupled from the others. For example, the n^{th} order term in W^i consists of a sum of the n^{th} order harmonic functions of $\nabla . LW = 0$ plus extra terms uniquely defined by lower order multipoles of ϕ and W^i. Hence, the far field is completely and uniquely defined by the harmonic functions of ∇^2 and $\nabla . L$.

Each multipole is physically identifiable. For example, the monopole term of ϕ is the mass. Also, the surface integral form of the momentum is[2]

$$16\, P_\xi = -2 \oint_\infty \sqrt{g}\, (LW)^{ij}\, \xi_i\, dS_j$$

where ξ_i is a translation or rotation vector at infinity, for momentum and angular momentum, respectively. One can show that the coefficients of the three first-order harmonic functions of W^i define the linear momentum and that the coefficients of three of the nine second-order harmonic functions define the total angular momentum.[11] The other multipoles can be understood by appealing to the Newtonian analogs of our equations. Many of their properties are analogous to those of their Newtonian counterparts, appropriately corrected for non-linearities. Most multipoles will not be constants of the motion. For instance, in our approach, with a reasonable definition of time,[12] one finds

$$\partial/\partial t\, [\text{dipole moment of } \phi] = [\text{linear Momentum}]$$

just as expected.

As we mentioned earlier, this generalized potential technique not only helps us analyze gravitational fields but also permits us to construct them. A constructive approach to Einstein's equations can proceed as follows. An initial metric is chosen and the related transverse - tracefree part of the extrinsic curvature is constructed;[9] these objects characterize the gravitational degrees of freedom.[13] The trace of the extrinsic curvature $K(x)$ is specified. This is an essentially kinematical datum that describes the initial slice.[13] The initial configuration of matter or fields, if any, is specified.[2,3] Then the elliptic equations are solved for ϕ and W^i, with the boundary conditions $\phi \to 1$, $W^i \to 0$ at spatial infinity. Having solved for ϕ and W^i one then has a complete and physically interpretable set of Cauchy data. The remaining Einstein equations give the time evolution of the initial state. The integration of these equations can also proceed via a numerical computer solution.[4] The only other steps necessary to carry this procedure through involves the specification of a spacetime coordinate system.[14] While the spacetime does not depend upon the choice of coordinates, it is natural to choose them so that the independent data has compact support on each hypersurface. An effective method for constructing such coordinates is to specify $\dot{K}(x,t) = 0$ and to require that[12]

$$\nabla^j \left(\dot{g}_{ij} - \tfrac{1}{3} g_{ij} g^{mn} \dot{g}_{mn} \right) = 0$$

The first condition leads to a linear elliptic equation for the lapse function N, the orthogonal proper time between two slices of the foliation; the second condition leads to a linear elliptic equation (in terms of $\nabla_i L$) for the vector field N^i describing the shift of the three-dimensional coordinates from surface to surface. This choice of shift can be shown from a simple variational principle[15] to minimize the shear of the gravitational field when computed along the four-vector $\partial/\partial t$. This choice minimizes coordinate effects in the evolution of g_{ij} and K_{ij} as much as possible.

Therefore, the resolution of the initial value problem by means of the potentials ϕ and W^i has opened up two new ways of increasing our understanding of the gravitational field. Firstly, it presents us with workable equations to construct,numerically or by other approximation methods, interesting solutions which cannot be studied using analytic techniques. Secondly, it permits us to characterize and analyze the asymptotic spacelike structure of gravitational fields, in terms of energy, linear and angular momentum, and multipole moments.

References

1. A foliation (of codimension one) is defined by a one-form $\underset{\sim}{u}$ with the properties $\underset{\sim}{u} \wedge \underset{\sim}{d}\underset{\sim}{u} = 0$ and $g(\underset{\sim}{u}, \underset{\sim}{u}) = -1$, where the spacetime metric has signature $(-+++)$.

2. N. Ó Murchadha and J. W. York, Phys. Rev. D10, 428 (1974); ibid., 10, 437 (1974). These articles contain references to our earlier work.

3. N. Ó Murchadha and J. W. York, J. Math. Phys. 14, 1551 (1973).

4. L. Smarr is applying these equations in the numerical investigation of the collision of two black holes (personal communication).

5. A detailed investigation of gravitational energy is given by the authors in Phys. Rev. D10, 2345 (1974).

6. See, for example, the article by R. Arnowitt, S. Deser, and C. W. Misner in Gravitation, edited by L. Witten (Wiley, New York, 1962).

7. D. R. Brill, Ann. Phys. (N.Y.) 7, 466 (1959).

8. Exact expressions are given in Ref. 2.

9. J. W.York, J. Math. Phys. 14, 456 (1973); Ann. Inst. Henri Poincare 21, 319 (1975).

10. Since the three-slice is Euclidean at infinity, there exists a quasi-Cartesian coordinate system in the far-field, and we define r with respect to this coordinate system. It can be precisely defined by the monopole harmonic function of ∇^2 on the manifold.

11. N. Ó Murchadha and J. W. York, "Gravitational Mass, Momentum, and Multipoles", to be submitted for publication.

12. This particular choice of spacelike hypersurfaces and time four-vectors is discussed in Ref. 11; also J. W. York and N. O Murchadha, Bull. Amer. Phys. Soc., April, 1974; J. W. York and L. Smarr, Bull. Amer. Phys. Soc., April, 1975; L. Smarr and J. W. York, paper to be published.

13. J. W. York, Phys. Rev. Lett. 26, 1656 (1971); ibid. 28, 1082 (1972).

14. One also has inevitable problems with numerical instabilities, as found by B. DeWitt, L. Smarr, and K. Eppley. These problems are being overcome using again the techniques the authors have developed in the initial-value analysis (L. Smarr and K. Eppley, personal communication).

15. L. Smarr and J. W. York, Ref. 12.

Biographical Sketch of Niall Ó Murchadha

Niall Ó Murchadha was born in Cork, Ireland on October 27, 1946. He received a B.Sc. degree from University College, Cork, and a Ph.D. degree in Physics from Princeton University. He was a Research Associate at Princeton University from February to August 1973. Since September 1973 he has been a Research Associate in the Department of Physics and Astronomy, University of North Carolina, Chapel Hill. He is married, with no children.

Biographical Sketch of James W. York, Jr.

James W. York, Jr. was born in Raleigh, North Carolina on July 3, 1939. He recieved the B.S. and Ph.D. degrees in physics at North Carolina State University, where he was a Ford Foundation Fellow. He served there as a member of the faculty for three years. He was selected to participate in the 1967 Battelle Rencontres in Physics and Mathematics. In 1968 he came to Princeton University as a Research Associate. He was awarded a National Science Foundation Postdoctoral Fellowship for 1969-1970. He was an Assistant Professor of Physics at Princeton from 1970-1973. Since August 1973, he has been an Associate Professor of Physics at the University of North Carolina, Chapel Hill. He is married to the former Betty Mattern of Salem, Virginia. They have two children, Virginia, 9, and Matt, 7.