THE ASYMPTOTIC SYMMETRY GROUP OF GENERAL RELATIVITY

Patrick J. McCarthy
University of British Columbia, Vancouver 8, Canada

Summary

It is suggested that, in a quantum theory which takes gravity into account, isolated quantum mechanical systems should be described by unitary representations of the asymptotic symmetry group of general relativity. In particular, elementary particles should be described by irreducible representations. It is pointed out that, in contrast to the conventional non-gravitational scheme, the particles all necessarily have discrete spins and a finite number of polarisation states. This agrees with observation. A physical picture of the resulting particles is given. The possibility of using the scheme to account for internal symmetries is discussed.

In specially relativistic quantum theory, the identity component of the isometry group of flat space-time, the Poincaré group, plays a fundamental rôle. The reason for this is as follows. According to quantum mechanics, each isolated quantum mechanical system is described by the projective space of one dimensional subspaces of a Hilbert space, and the measurements that can be made on the system, the transition probabilities, are given in terms of the modulus of the scalar product of vectors in the Hilbert space. The content of the "principle of relativity" in such a setting is that the transition probabilities are invariant under Poincaré transformations. Using only these very general assumptions, it can be shown[1-3] that every isolated quantum mechanical system must be described by a unitary representation of the Poincaré group in the Hilbert space associated with the system. In his classic paper, Wigner[1] classified all such representations, thus obtaining a complete description of all possible such quantum mechanical systems. The classification corresponds to a classification of all possible specially relativistic wave equations.[1-5] Actually, the representation description is superior to the wave equation description, since the former yields directly the entire space of wave functions describing the system, together with their transformation properties, whereas the latter merely gives a partial differential equation to be solved for a dense subset of differentiable wave functions, with no transformation properties. (However it seems that interactions are easier to introduce in the latter approach, but see reference 6 for a group theoretical approach to the interaction problem). Another advantage of the representation approach is that it provides, perhaps, the clearest definition of an "elementary particle" known. Indeed, if a quantum mechanical system is such that it has no Poincaré invariant proper subsystems, it surely represents one's intuitive idea of an elementary particle, so that one makes the definition: (A) <u>An elementary particle is (a quantum mechanical system associated with the representation space of) an irreducible representation of the Poincaré group.</u>[2,4]

This definition brings out very clearly that the significant space-time parameters which distinguish elementary particles are the <u>mass squared</u> and the <u>spin</u>. Setting aside until later the question of "internal" symmetries, one may justifiably say that Wigner's approach and the developments it gave rise to provide the most successful description to date, philosophically and practically, of free relativistic particles. Nevertheless, the approach has several limitations. For example, one can criticize (A) on a point of principle: 1(A) According to general relativity, space-time is not flat, and any reasonably general space-time has no isometries whatsoever. In particular, the Poincaré group is not defined. One can also raise the following practical objections: 2(A) The definition allows the mass squared to take any real value, that is, it allows a continuous mass spectrum (the observed mass spectrum is discrete), and 3(A) the definition allows three different types of spin: discrete spins with a finite number of polarisation states, continuous spins with an infinite number of polarisation states, and discrete spins with an infinite number of polarisation states (only the first type is observed).

In this essay, I wish to present arguments in favour of a new scheme which appears to overcome some of the above objections. Both in quantum theory and in general relativity, many people have for some time concentrated their efforts on studying the asymptotic properties of physical systems. For example, the basis of S-matrix theory is that, since only the asymptotic states in scattering experiments can in principle be observed, the theory should only be concerned with asymptotic properties. Again, in general relativity, the study of bounded sources emitting gravitational radiation is carried out mainly by analysing the behaviour of the fields asymptotically in light like directions. This makes sense observationally, since the information which an outside observer collects is precisely of the type that the theory analyses. This work in gravitational asymptotics began in an important paper by Bondi, Metzner and van der Burg,[7] and was generalised by Sachs.[8] These authors studied a class of asymptotically flat space-times representing the gravitational systems referred to above, and their work has been followed up and generalised by many other people. From the present point of view, the major result of these authors was as follows: They found that the group of asymptotic isometries (transformations defined in a neighbourhood of "infinity," which approach isometric ones sufficiently near to "infinity") was independent of the details of the system, and that it contained, but was larger than, the Poincaré group. The group, now called the Bondi-Metzner-Sachs group (BMS group), may be thought of, roughly, as the group which preserves the asymptotic flatness of asymptotically flat space-times. This BMS group immediately attracted attention as a possible candidate for replacing the Poincaré group in a microphysics which included gravity, or as an important tool in the problem of quantising the gravitational field.[9-11]

Here I would like to be a little more specific. Thought of classically, an elementary particle in an otherwise flat space-time may be represented, as far as its gravitational properties are concerned, as an asymptotically flat space-time. In practice, of course, the ambient space-time is not flat (it has cosmological curvature or, more provincially, curvature due to the earth's gravitational field). However, the ratio of the space-time curvature due to the particle at its surface to the space-time curvature due to the earth is of the order of the ratio of the mass densities of the particle and the earth, so that it is an excellent approximation to represent the particle as an asymptotically flat space-time.[12] Since experimental information about the particle is necessarily asymptotic, it seems eminently reasonable to describe the particle entirely in terms of asymptotics. But then, though no exact isometry group need be defined, the asymptotic group is well defined. In view of the remarks in the preceding paragraph, it seems very plausible that one could achieve a fusion of quantum theory and general relativity at the purely asymptotic level by making the following definition: (B) an elementary particle is an irreducible unitary representation of the Bondi-Metzner-Sachs group. This gives rise to the definite mathematical problem of finding these representations, which was initiated by Sachs[9] and Cantoni.[13-15] I recently studied this problem systematically by means of Mackey theory,[16] and found all of the irreducible unitary induced representations of the group,[17,18] and examined, with M. Crampin, their restriction to the Poincaré subgroup.[19] It was shown that these representations all have "little groups" which are compact[18] (unlike those of the Poincaré group). This has the consequence that the BMS group only admits spins of the first type referred to in objection 1(A), discrete spins with a finite number of polarisation states - the observed type. Thus it would appear that definition (B) answers objections 1(A) and 2(A). However, it certainly does not answer 2(A).

From a study of the representations, one can get some idea of what these elementary particles look like physically. Classically, imagine the localised system as surrounded by a large sphere (celestial sphere) on whose surface the radiation information from the centre registers (in practice, one could think of the sphere of having a radius of $\sim 10^{-1}$ cm>>10^{-13} cm). The generalised momentum (or "supermomentum"[17,18]) of the system is simply given by some distribution of radiation on the sphere, the "little group" being the symmetry group of this distribution. With this picture, one can already understand intuitively why the little groups of the BMS group all have to be compact.[20] The spin of the system simply corresponds to the rotational symmetries of the radiation distribution, so that particles with distributions of low (high) symmetry have simple (complex) types of spin.

As far as "internal" symmetries are concerned, there are, in addition to the little groups corresponding to the directly observed types of Poincaré spin, several others, and it is possible that some of these may be interpreted in terms of "internal" symmetries.[18,19,21] This would be attractive, since the internal symmetries would then have a direct geometrical interpretation. On the other hand, if it proved to be necessary to combine conventional internal symmetry groups with the BMS group, then O'Raifeartaigh's proof,[22] for the Poincaré group, of the impossibility of mass splitting, no longer holds[17,21] (because the BMS group is infinite dimensional), so this approach holds out some hope of removing 2(A). Thus definition (B) does seem to hold considerable promise in some respects. However, much remains to be done on the problem - for example, to obtain wave equations and introduce interactions probably requires a better understanding of purely asymptotic physics. My latest work on the problem has been on lifting of projective representations to true ones (c.f. reference 2), a problem I have almost, but not quite solved. I certainly remain convinced that asymptotic symmetries will continue to play an important rôle in physics in the future.

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Biographical Sketch

The author did undergraduate work in physics and mathematics at Oxford and Cambridge, and was strongly attracted towards relativity, and quantum theory. As a research student at King's College, London, he also became interested in symmetries in theoretical physics, and was delighted to find a Ph.D. topic, the B.M.S. group, which combined all interests at once. He now holds a postdoctoral fellowship at the University of British Columbia, and is moving to Dublin in a few months. He is married and has a small son.