AN INTERACTING GEOMETRY MODEL AND INDUCED GRAVITY

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Abstract

We propose the theory of quantum gravity with interactions introduced by topological principle. The fundamental property of such a theory is that its energy-momentum tensor is an BRST anticommutator. Physical states are elements of BRST cohomology group. The model with only topological excitations, introduced recently by Witten is discussed from the point of view of induced gravity program. We find that the mass scale is induced dynamically by gravitational instantons. The low energy effective theory has gravitons, which occur as the collective excitations of geometry, when the metric becomes dynamical. Applications of cobordism theory to QG are discussed.

The idea that topology of spacetime must play the fundamental role in any reasonable theory of Quantum Gravity (QG) was first put forward by Wheeler[1]. This highly original physical proposal was subsequently taken seriously by a number of people [2,3,4], especially after the period of intensive studies of nonperturbative properties of gauge theories (instantons, monopoles). The concept of gravitational instanton was introduced by Hawking[4]. The classical role of instantons is to break classical symmetries of the action functional for a given physical system. A beautiful example of this phenomenon is the t'Hooft solution[12] of the chiral symmetry breaking problem in gauge theories (the famous $U(1)$ problem). We will see that this concept will play an important role in our proposal of the QG model whose only degrees of freedom are the global, topological degrees of freedom. The model we propose in this essay shares many qualitative properties with the modern theory of strings[6].

One may ask how we would expect to recover local physics, and in particular classical General Relativity (GR), from a model whose only degrees of freedom are the global, topological characteristics of the four-manifolds? We do not have the definite answer to this question yet, but it is clear that the four-metric on a topological four-manifold becomes a dynamical degree of freedom at an energy scale M_P, which is of the order of the Planck mass. Therefore, if our model displays the property of dynamical scale symmetry breaking like Quantum Chromodynamics (QCD), then the dimensional transmutation phenomenon operative in QCD should take place. The dimensionless unrenormalized coupling constant g_0 which parametrizes our model gets replaced by the renormalization group invariant mass parameter $M_P(g, \mu)$, which will play the role of the Planck mass in the low energy effective theory of gravity (GR).

Below the Planck energy, the model of QG is described by an effective low energy theory. This low energy effective theory, which is generally covariant and causal in the classical limit, must be described by a lagrangian at most quadratic in the first derivatives of the metric. There is only one such lagrangian, the Einstein-Hilbert lagrangian of General Relativity with a possible cosmological constant. The basic property of some of the QG models we will discuss in this essay is that the metric becomes dynamical as a result of an instanton induced phenomenon. The mechanism which is operative in the models we discuss here is in the spirit of the induced gravity program initiated over twenty years ago by Zeldovich[2] and Sakharov[3], and later pursued by Adler and his colleagues[9,11].

Quantum fluctuations of geometry and topology become important at the Planck scale, $L_P = (Ghc)^{1/2} = 10^{-33} cm$. The argument showing the plausibility of Wheeler's conjecture(assertion) is based on the Eistein-Hilbert action or any theory of gravity with the fundamental mass scale, either explicit in the action or dynamically induced. It can be easily seen, using the Feynman path integral approach and dimensional arguments, that the probability for topology changing on distances larger than L_P is enormously suppressed, being of order $e^{-(L/L_P)^2}$, where L is the scale on which topology of the four-manifold changes. Therefore, in the classical dynamics of the gravitational field, we do not expect dynamical change of topology of spacetime; toplogy of the 4-manifold *is not the dynamical degree of freedom* of classical general relativity (GR). Before going to the more specific discussion of the topological aspects of Quantum Gravity (QG) let us first discuss in general terms those properties of GR which are common with string theory.

Classical general relativity is the theory of dynamics of three-geometries, and the (closed) string theory is the theory of dynamics of one-dimensional geometries- strings. In the hamiltonian formulation of GR (string theory) we specify the topological three-(one-)manifold Σ and a collection of fields on it. For GR this is the three-metric q_{ab} and its conjugate momentum π^{ab}; for the string theory in the Polyakov formulation it is instead the embedding scalar field X^{μ} and the conjugate momentum Π_{μ} (for simplicity we will assume Σ to be compact without boundary). The canonical variables are specified modulo the large gauge symmetry group, $Diff(\Sigma)$. As usual with hamiltonian systems with large local symmetry groups, the symmetry group $Diff(\Sigma)$ is generated by constraints H_a, which form (classically) a closed algebra isomorphic to the algebra of $Diff(\Sigma)$. The classical dynamics is defined by the hamiltonian H_0, which vanishes in reparametrization invariant (covariant) theories like GR or string theory. H_0 evolves $\Sigma_i = \Sigma$ to $\Sigma_f = \Sigma$, and the "history" of the geometry Σ is the manifold $M = \Sigma \times R$, on which the particular form of constraint algebra induces the Lorentz structure. In the case of three-geometry, we recover in this way the generally covariant Einstein-Hilbert action for GR. Of course, historically this was understood the other way around.

Consider the "history" M, and a function t on it. Smooth and regular function t, which can be called "local time" defines a one-parameter family of slices $\Sigma(t)$ in M. In fact, t is a Morse function on M and its critical points carry information about the topology of M. If M is a product manifold $\Sigma \times R$, then t has no critical points (i. e. points) where $dt = 0$, and starting with an arbitrary, smooth riemannian metric, using $grad(t)$ one can construct causal Lorentz[13] structure on M. When t has critical points on M as happens for the "trousers" topology, with $\partial M = \Sigma_i \cup \Sigma_f$, where $\Sigma_i = \Sigma_1$, and $\Sigma_f = \Sigma_2 \cup \Sigma_3$, then the hamiltonian dynamics is not well defined because the hamiltonian H_0 "does not know" how to evolve the geometry Σ_1 beyond the splitting region; it vanishes identically at the critical points of the "time" function.

It is a classical result due to Geroch[10] that nontrivial cobordisms does not admit a smooth causal Lorentz structure on the four-manifold. Closed timelike, or null curves must occur on manifolds with the topology of "trousers". Similar phenomenon occurs on two-dimensional cobordisms (Sorkin, private communication). Classical evolution of matter or gravitational fields is not well defined on nontrivial cobordisms, and similarly, quantum field theory(QFT) in such backgrounds is not well defined[8]. However, this does not mean that nontrivial cobordisms are not important in QG. Nontrivial topologies may play a fundamental role in QG, where we have to recognize the importance of quantum fluctuations. To put it strongly, we shall propose in this essay that all topologies of manifolds in $D = 2, 3, 4$-dimensional QG must be considered[7]. This might be necessary in order to have a unitary and, presumably well defined, QG at high energies.

Let us recall here the definition of an (un-)oriented cobordism. Two manifolds are called cobordant if their disjoint union (modulo extra structure like orientation, spin structure,complex structure etc.) bounds a closed smooth manifold called the cobordism i. e. $\partial M = \Sigma \cup \Sigma'$. The last relation is an equivalence relation between closed manifolds which transforms the space of closed manifolds into an abelian cobordism group Ω_d, with disjoint union as the group operation.

The trivial elements of this group are all manifolds which are boundaries. For our purpose it is important to know that $\Omega_1 = \Omega_2 = \Omega_3 = 0$. This means that, at least in the path integral approach to QG, there are no topological obstructions in defining a quantum mechanical amplitude for the d-geometry in the "state" with the topology Σ_i to evolve to the "state" with topology Σ_f. There always exist "histories" (cobordisms) interpolating between initial and final "states" $\Sigma_{i,f}$ in $D = 2, 3, 4$-dimensional QG.

To put the basic idea to work, let us now consider the instructive case of $D = 2$ QG (or string theory). Any 1-dimensional compact manifold Σ is simply a disjoint union of circles S^1. There always exists a cobordism M (string world-sheet) whose boundary is Σ. However, this cobordism is not unique unless we require that M be simply connected (one can always have "holes" in M). In fact, requiring M to be simply connected, we realize that the simplest cobordism is a sphere S^2 with a number of discs D^2 removed; otherwise M is an arbitrary Riemann surface with a number of discs removed. However, there always exist elementary cobordisms from which we can construct an arbitrary cobordism by "gluing" together elementary cobordisms (we can also construct in this way all compact manifolds without boundary, for $D = 2$). We find that the elementary cobordisms in $D = 2$ QG are the spheres with two and three discs removed. In closed string field theory these are called the propagator and vertex, respectively. Interaction in string field theory is introduced by allowing for nontrivial cobordisms (string vertex).

The quantum mechanical amplitude A for quantum geometry is defined as a mapping between the cobordism $M : \partial M = \Sigma_i \cup \Sigma_f$ with prescribed boundary conditions $\Phi_{|\partial M}$ on physical fields Φ on the boundary $\Sigma_i \cup \Sigma_f$ and complex numbers. The boundary conditions select a particular state from the "asymptotic" (Fock) Hilbert space for each disjoint component of $\Sigma_{i,f}$.

$$A[M, \Sigma_{i,f}, \Phi_{|\partial M}] = \int D\Phi e^{-I[\Phi]}, \tag{1}$$

where Φ is the collection of fields depending on the model. In string theory $\Phi = (g, X)$, where g_{ab} is the world-sheet metric and X^μ is the scalar embedding field. In $D = 4$ QG, the collection of fields Φ should include the 4-metric $g_{\alpha\beta}$ as well as other (matter) fields.

String theory is the simplest, but complicated enough, example of a theory in which the interaction is introduced by the *topological principle.* Witten[6] has constructed a quite ingeneous string field theory for open strings. The cornerstone of his construction is the BRST formulation of the string theory on a given cobordism (string diagram). Witten starts his construction with the gauge-fixed string action i. e., with an action which is not reparametrization invariant. However, this new action includes Grassmann fields (Faddev-Popov ghosts) and has a fermionic symmetry—BRST invariance. When studying given theory in the BRST formulation, one has to bear in mind that *BRST invariance is a substitute for reparametrization invariance.* BRST invariance implies that there is a conserved current whose charge Q is nilpotent i. e., $Q^2 = 0$. The crucial property of gauge-fixed string action is that the "energy-momentum" tensor $T_{\alpha\beta}$ is the BRST anticommutator

$$T_{\alpha\beta} = [Q, b_{\alpha\beta}]_+, \tag{2}$$

where $b_{\alpha\beta}$ is the anti-ghost field. This implies that the expectation value of $T_{\alpha\beta}$

vanishes on physical states:

$$\langle T_{\alpha\beta} \rangle = 0. \tag{3}$$

This is the most important equation, because it implies that, even if the metric $g_{\alpha\beta}$ enters explicitly the action, the quantum mechanical amplitudes do not depend on the metric of a given (string)cobordism. Also, the commutator of Q and the ghost number U is: $[U, Q] = Q$. A state is called BRST invariant if Q anihilates it: $Q\psi = 0$. The most interesting solutions of the last equation are those that cannot be written in the form $Q\lambda$. The equivalence classes of solutions of $Q\psi = 0$ with a given ghost number form the BRST cohomology groups. Physical states of the BRST quantized system have definite ghost number U_0, which depends on the particular theory. We conclude that physical states form the BRST cohomology group H^{U_0}. Witten's theory[6] is cubic in the string field operators. This fundamental property of Witten's theory corresponds to the basic observation derived from cobordism theory; there exist only two elementary cobordisms in string dynamics: the two-holed sphere corresponding to the bare propagator which defines the kinetic part of the second-quantized action for string fields, and the three-holed sphere (string vertex) corresponding to the basic cubic interaction term in the action.

Encouraged by the Witten's path breaking work, some time ago[7] we proposed a model of QG similar to Witten's string field theory. However, this was only the suggestion that one should seriously consider the possibility of interacting three-geometries with different topologies as the fundamental principle for construction of a Quantum Theory of Gravity. Our model of QG is one in which the interaction is introduced by the topological principle. In analogy with string theory we conjecture that $D = 3$ QG is cubic in the wave functional of the three-geometry. The elementary $D = 3$ cobordisms are three-manifolds with two or three boundaries, which are arbitrary Riemann surfaces. The fundamental object whose dynamics we will study in $D = 4$ QG is the three-geometry Σ. And here is where all the problems seem to start. Unlike string theory, where the basic object is the one-manifold, S^1, in QG_4 there are a plethora of possibilities for the topology of a three-manifold. There exists an infinite number of different topological three-manifolds, which can presumably allow for a complete classification. What makes QG very difficult to study on the formal mathematical level is this incredible richness of basic objects. Another problem is our relatively poor understanding of the local and global properties of the gauge group of QG, the diffeomorphism group $Diff(\Sigma)$.

However, despite the existence of all these possible topologies Σ in QG, it is possible to construct a model whose quantum dynamics is independent of the metric. It is sufficient to choose the matter plus ghost system in such a way that the energy-momentum tensor is a BRST anticommutator. Then we study quantum mechanical amplitudes given by the the path integral (1), where Φ is a collection of matter fields, ghosts, and the metric. If the fundamental (gauge-fixed) action is invariant under a BRST-type symmetry with the energy momentum tensor in the form of a BRST anticommutator, then the amplitudes $A[\Sigma_i, \Sigma_f, \Phi|_{\partial M}]$ will be independent of the particular metric chosen on the cobordism. This means that the path integral over metrics factorizes, yielding an irrelevant infinite constant. Assume that somehow the generator of fermionic symmetry does not annihilate a certain state. Then the expectation value of the energy-momentum tensor will be non-zero and and it will depend explicitly on the metric chosen on the cobordism. In principle, we can integrate the equation defining the variation of the effective action with respect to the metric,

$$\delta\Gamma = 2\int d^4x\sqrt{g}\delta g^{\alpha\beta}T_{\alpha\beta}, \tag{4}$$

to obtain the induced action for gravity $\Gamma[g]$. The question we have to ask is: does there exist a mechanism which can be responsible for the spontaneous or dynamical symmetry breaking of the fermionic symmetry Q ? This depends on the model. Witten has recently[5] constructed a model with fermionic symmetry Q such that the quantum mechanical amplitudes defined by cobordisms depend only on the topological invariants introduced by Donaldson in his studies of the theory of four-manifolds. Witten's model is probably not very physical, but it gives a simple physical interpretation for Donaldson's topological invariants. From our point of view, Witten's model has the attractive property that the Q symmetry is broken by gravitational instantons (closed 4-manifolds without boundary) i.e. by vacuum fluctuations.

Let us describe the argument briefly, without going into the details of Witten's model. The argument is based on Witten's index $Tr(-1)^F$, familiar from supersymmetric theories. The standard argument due to Witten shows that, if this index vanishes, then fermionic symmetry (supersymmetry) must be spontaneously or dynamically broken. The operator $(-1)^F$ anticommutes with fermionic charges, $[(-1)^F, Q]_+ = 0$. Does there exist an operator in Witten's model[5] which has these properties? The answer is yes. This is the operator $e^{i\pi U}$, where U is the ghost number operator, which is defined modulo 8 for an $SU(2)$ gauge theory with fermionic symmetry. One can show that the generalized Witten index for this model[7] is zero if the so-called formal dimension

$$d(M) = 8p_1(E) - \frac{3}{2}(\sigma(M) + \chi(M)), \tag{5}$$

of the moduli space of selfdual $SU(2)$ gauge connections is nonzero. $p_1(E)$ is the first Pontryagin number of the $SU(2)$ bundle E, and χ and σ are the Euler characteristic and signature of M. We conclude that there exist gravitational, and at the same time Yang-Mills instantons (one can call them "mixed" instantons) which break the fermionic symmetry Q. An example of such an instanton is the familiar $CP(2)$ instanton. $\chi + \sigma = 2B^{+}{}_2$, where $B^{+}{}_2$ is the number of selfdual closed two-forms on M. One can construct a number of such instantons. Instantons break the apparent symmetry of the classical action. This phenomenon is quite similar to the problem of chiral symmetry breaking in QCD. What is the physical meaning of this phenomenon? It simply means that the theory which is generally covariant displays the property of dynamical symmetry breaking. Witten's model is scale invariant, because it is reparametrization invariant. Scale symmetry is dynamically broken and the theory acquires a mass scale. At the same energy scale the metric becomes a dynamical degree of freedom. We end up with induced gravity as a low energy effective theory of gravity. Much more work need to be done before we will be able to understand properties of theories with the topological interaction principle. However,one thing seems to be clear, namely that the idea of finite, nonlocal gravity beyond the Planck scale may shed some light on the issue of calculability of amplitudes in QG.

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