SUMMARY

A theorem is proven that the only possible exterior solution for a psuedo-stationary, rotating, electrovacuum black hole with non-degenerate event horizon is the Kerr-Newman solution with $M^2 - J^2/M^2 - Q^2 > 0$. A special role played in the proof of this theorem by the hidden symmetry group $SU(1,2)$ of Einstein's equations is established

# Black Hole Uniqueness from a hidden Symmetry of Einstein's Gravity

Essay on Gravitation.

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## Introduction

The question of uniqueness of an asymptotically flat black hole equilibrium state, in the absence of external sources received some attention recently. The formation of a black hole is an essentially stabilizing effect. An object collapsing from a general initial configuration will settle down into one of possible black hole stationary equilibrium states. The only multipole moments which would remain as independent degrees of freedom characterizing the black hole equilibrium state, would be those which do not couple to corresponding degrees of freedom in the asymptotic radiation field. The remaining multipoles will be radiated away.

It was conjectured therefore, that stationary isolated black holes can be characterized completely by three parameters M, J, Q which represent asymptotically conserved total mass, angular momentum and electric charge, respectively. This conjecture has been proved only quite recently.

Do there exist non-Kerr-Newman black hole equilibrium states ? The answer to this question is of great importance for relativistic astrophysics. If they exist, they would have the anomalous property that the quasi-stationary classical spin-down and discharge processes cannot be carried out completely. It would seem reasonable to conclude, therefore, that non-Kerr-Newman black hole equilibria do not exist. The purpose of this essay is to show that they do not exist indeed.

## The Newtonian analogy

Perhaps one of the most important results on stationary black hole properties is the Hawking strong rigidity theorem. According to this theorem the domain of outer communications of an asymptotically flat, stationary black hole should be either <u>non-rotating</u> or <u>axisymmetric</u> provided that the vacuum Einstein or Einstein-Maxwell equations are satisfied. The first result in the non-rotating static case was obtained by Israel [1] who has shown that the static black hole is the Schwarzschild one or its electrovacuum generalization - a result rather surprising at the time when it was presented.

One can show that in the axisymmetric case the domain of outer communications must have <u>circular symmetry</u> i.e. the isometry group has to act orthogonally transitively on it. Under the last condition the source-free Einstein-Maxwell equations can be expressed as a non-linear, elliptic system of partial differential equations for two complex Ernst's potentials [2,3] $\epsilon, \psi$. The Ernst approach to the field equations allows for an extremely simple formulation of the global black hole boundary conditions.

The question of uniqueness for the black hole boundary problem seems to be difficult to answer. It is a general property of nonlinear elliptic equations that bifurcations do take place [4].

In order to gain insight into how the problem should be approached, consider first an analogous problem in the Newtonian theory of gravitation. The gravitational potential $\varphi$ is a solution to the global boundary value problem for Laplace's equation

$$\nabla^2 \varphi = \Delta \varphi = 0 \quad . \quad (1)$$

In this case the question of uniqueness is solved by means of the Green identity. Let $\varphi_1$, $\varphi_2$ be two solutions to the boundary value problem of Laplace's equation. Then for the field $\varphi = \varphi_1 - \varphi_2$ we have the Green identity

$$\nabla^2\left(\frac{1}{2}\varphi^2\right) = \nabla\cdot(\varphi\nabla\varphi) = \varphi\nabla^2\varphi + \nabla\varphi\cdot\nabla\varphi \quad . \quad (2)$$

The first term on the r.h.s. vanishes, the second one is nonnegative. The divergence term on the l.h.s. , when integrated over the whole domain, gives the surface integral which vanishes because of boundary conditions. One sees then, from the nonnegativity of the r.h.s. , that the two solutions must be identical.

Is there a deeper reason for the existence of Green's identity ? The structure of this identity suggests, that it is not the special form of the field equation, but its more general properties which are important.

The Laplace equation is the simplest example of what is known as a harmonic map [5] equation. The harmonic map between two Riemannian manifolds $(M, g_{\mu\nu})$ and $(M', h_{AB})$ with coordinates $x^\mu$ on M and $\phi^A$ on M' exstremizes the action functional

$$S = \frac{1}{2}\int\sqrt{g}\, dx\, g^{\mu\nu}\phi^A{}_{,\mu}\phi^B{}_{,\nu}\, h_{AB}(\phi) \quad . \quad (3)$$

In the case of Laplace's equation (1) we have a mapping of the three-space [illegible] into the Euclidean line [illegible]. The global inner symmetry [illegible] is the group of translations, $\phi \to \phi' = \phi + c$. One can see, that this is a symmetry of the Green identity, too. The l.h.s. of Green's identity is the Laplacean of the Euclidean invariant $\sigma = \frac{1}{2}s^2 = \frac{1}{2}(\varphi_1 - \varphi_2)^2$, where $s(\varphi_1, \varphi_2)$ is the bi-scalar of the global distance between two points on $E^1$.

Let us now turn to the case of the Einstein-Maxwell equations. Guided by the analogy with the Newtonian theory of gravitation, we shall present the argument that an identity analogous to the Green's one should exist for Einstein's theory.
D.C.Robinson gave an elegant and siple proof [7] of uniqueness of the Kerr black hole solution. His proof was based on the application of a nonlinear identity, which fortunately turns out to exist for Einstein's vacuum equations.

It is well known [6] that Einstein and Einstein-Maxwell equations for stationary fields possess hidden global symmetries, $SU(1,1)$ and $SU(1,2)$ for the vacuum and electrovacuum cases, respectively.

One can easily (but labouriously) check that the nonlinear Robinson identity is invariant under the nonlinear, homographic representation of the symmetry group $SU(1,1)$. However, even if this is a key observation, one cannot guess simply how to construct the analogou electrovacuum identity. The main obstacle is the essentially nonlinear way in which the $SU(1,2)$ group acts on field variables.

Let me put things somewhat differently. If we were able to "linearize" the inner symmetry group action and transform the fiel.

equations to a form covariant under linear representation of the symmetry group, we would expect the required electromagnetic identity appear naturally.

The reduced system of Einstein-Maxwell equations is specified by the variational principle with non-negative Lagrangian (5,8)

$$L = \frac{1}{2}\,\frac{(\nabla\mathcal{E} + 2\bar{\Psi}\nabla\Psi)\cdot(\nabla\bar{\mathcal{E}} + 2\Psi\nabla\bar{\Psi})}{X^2} + 2\,\frac{\nabla\Psi\cdot\nabla\bar{\Psi}}{X} \qquad (4)$$

where $\mathcal{E} = -X-E^2-B^2+iY$, $\Psi = E+iB$ and $\nabla$ is a covariant derivative operator with respect to a fixed metric on a 2-space V orthogonal to the isometry group orbits. $(X,Y)$ and $(E,B)$ are gravitational and electromagnetic potentials, respectively.

One can show that the Lagrangian (4) defines a harmonic map between a 2-space V and the Kahler symmetric space $\mathcal{H} = SU(1,2)\,/\,S(U(1)\times U(2))$ i.e. a nonlinear $\sigma$- model on $\mathcal{H}$ (8,9). Since we want the action of the $SU(1,2)$ group on $\mathcal{H}$ to be linear, we would like to find a convenient parametrization $\Phi$ of it. There exists a fairly natural parametrization of a Riemannian symmetric space $\mathcal{H} = G/H$ in terms of a G-valued matrix field $\Phi$. The field equations, in this approach are given by $\nabla_\mu j^\mu = 0$, where $j_\mu = \nabla_\mu \Phi\,\Phi^{-1}$. A global symmetry group G acts on $\mathcal{H} = G/H$ from the left in a linear way. The field equations are covariant (8,9) under the linear representation of the inner symmetry group $G = SU(1,2)$.

The generalized electrovacuum identity, as well as the vacuum Robinson's one, appears naturally in this approach. Let us consider

..o fields $\Phi_1$ and $\Phi_2$ ... not necessarily solutions to the field equations. One can de... ... field $\Phi = \Phi_1 \Phi_2^{-1}$ which trans... ...ly under the ... ...formations of $\mathcal{H}$ as $\Phi' = u \Phi u^{-1}$, $u \in G$. ...r $\Phi$ is the requi... ...$(\cdot,\cdot)$ invariant. It plays the same role in Einstein's theory of gravitation as translational invari... $\sigma = \frac{1}{2}(\varphi_1 - \varphi_2)^2$ in the Newtonian gravity. With this choice of invariant we apply to it the covariant derivative $\nabla$ twice to obtain the identity (generalized Robinson identity)

$$\mathrm{Tr}\left\{\nabla\cdot(\rho\nabla\Phi) + \Phi\left[\nabla\cdot(\rho j^{(2)}) - \nabla\cdot(\rho j^{(1)})\right]\right\} =$$

$$= \rho\,\mathrm{Tr}\left\{\Phi\left[j^{(1)}\cdot j^{(1)} + j^{(1)}\cdot j^{(2)} - 2 j^{(2)}\cdot j^{(1)}\right]\right\} \quad . \quad (5)$$

The r.h.s. of (5) is nonnegative whenever the Riemannian symmetric space $G/H = \mathcal{H}$ has a nonpositive sectional curvature or, what is the same when the global symmetry group G is a non-compact one. This is the case for the Einstein theory of gravitation. The r.h.s. of (5) equals zero when $j^{(1)} = j^{(2)}$ i.e. when $\mathrm{Tr}\,\Phi = \mathrm{const}$.

In order to succesfully apply the global black hole boundary conditions we need only the form of $\mathrm{Tr}\,\Phi$ in terms of physical field variables $(X_i, Y_i, E_i, B_i)$, i=1,2. Since there exists a simple relation between $\Phi$-parametrization and the Ernst one,[8] we have

$$\mathrm{Tr}\,\Phi = 3 + X_1^{-1} X_2^{-1}\left\{(X_1 - X_2)^2 + 2(X_1 + X_2)\left[(E_1 - E_2)^2 + (B_1 - B_2)^2\right] + \right.$$

$$\left. + \left[(E_1 - E_2)^2 + (B_1 - B_2)^2\right]^2 + \left[Y_1 - Y_2 + 2(E_2 B_1 - E_1 B_2)\right]^2\right\} \quad . \quad (6)$$

the divergence term, when integrated over the whole domain turns to vanish, because of black hole boundary conditions. This leads to the conclusion that the two solutions must coincide.

## Summary

To summarize, under the condition that there is a non-degenerate event horizon i.e. $M^2 - J^2/M^2 - Q^2 > 0$, solution for the global black hole boundary problem does exist and the only possible one is the Kerr-Newman black hole solution.

The identity presented above has an interesting geometrical interpretation. One can show that the divergence term is of the form $\nabla^2 \sigma$, where $\sigma = \frac{1}{2} s^2$, $s(\phi_1, \phi_2)$ is the <u>bi-scalar of geodesic distance interval</u> which gives the magnitude of the invariant "distance" between $\phi_1$ and $\phi_2$ as measured along geodesics joining them. On the r.h.s. there are terms vanishing when $\phi_1$, $\phi_2$ are solutions to the field equations, as well as a quadratic form in $\nabla\phi_1$, $\nabla\phi_2$.

The key to the proof of uniqueness of the Kerr-Newman black hole solution is the observation that one can construct an identity, which originates from the hidden symmetry of Einstein's equations for <u>stationary</u> fields.

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