ON A NEW GRAVITATIONAL EFFECT OF A ROTATING MASS

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Summary. A new general relativistic many-body effect is described. It results in an unexpectedly large relative acceleration between neighboring test particles that follow an inclined orbit about a rotating mass. The effect vanishes if the orbit coincides with the equatorial plane of the rotating mass. The existence of this effect is due to a small divisor involving the deviation of the orbital frequency measured by a comoving clock from the frequency measured by an inertial clock. The influence of the rotation of the Sun on the Earth-Moon system is investigated, and it is shown that the new effect causes a harmonic variation in the Earth-Moon separation with an amplitude of order one meter and dominant periods of 18.6 years, $\sim 1/2$ year, 1 month and $\sim 1/2$ month. The confirmation of these results by the lunar laser ranging experiment would provide a significant new test of general relativity and a measurement of the angular momentum of the Sun.

The close analogy between Newton's law of gravitation and Coulomb's law led Holtzmüller to postulate the existence of a gravitational "magnetic" field in addition to the Newtonian "electric" field /1/. The precession of planetary orbits in the gravitational "magnetic" field of the Sun were calculated by Holtzmüller /1/ and Tisserand /2/, and attempts were made to reconcile the Newtonian theory of gravitation with the excess perihelion precession of Mercury on the basis of the new hypothesis. With the advent of general relativity, the excess perihelion motion of Mercury was explained by Einstein as being due to the post-Newtonian spherical field of the Sun. Moreover, Thirring and Lense /3/ investigated the gravitational "magnetic" field generated by a rotating mass and showed that the perihelion precession due to the rotation of the Sun is retrograde and much smaller than the post-Newtonian effect.

According to general relativity, the various effects associated with the motion of a particle in the field of a rotating mass may be attributed to the phenomenon of the dragging of the inertial frames. A rotating mass drags the local inertial frames around itself with respect to the inertial frame of the static observers at infinity, since the dragging frequency falls off rapidly with the distance away from the mass. In the lowest order of approximation, the dragging frequency reduces essentially to a gravitational "magnetic" field.

Let us now consider the relative acceleration of neighboring test particles in the field of a rotating mass. The result consists of the contribution of the spherical field of the mass M, together with the contribution of the higher moments. The dominant relative acceleration per unit separation of the test particles due to the angular momentum J may be estimated to be $\sim \mathcal{K}_o$,

$$\mathcal{K}_o = \omega_o^3 \frac{J}{Mc^2} . \qquad (1)$$

Here ω_o is the Keplerian frequency of the test particles around the rotating mass,

$$\omega_o^2 = \frac{GM}{r^3} , \qquad (2)$$

so that $2\pi / \omega_o$ is the period of the motion according to static observers at infinity. Based on estimate (1), the relative accelerations in the solar system due to the rotation of the Sun turn out to be negligibly small /4/.

A new and remarkable result is discovered, however, when one calculates the relative acceleration between neighboring test bodies to all orders in M and to first order in J. The dominant amplitude (per unit separation of test masses) turns out to be /5/

$$\sim \alpha \left(\frac{GM}{c^2 r}\right)^{-1} \mathcal{K}_o \qquad (3)$$

for a small inclination angle α , $\alpha << 1$, between the orbital plane and the equatorial plane of the rotating mass. For $\alpha = 0$, i.e. when the orbit is in the equatorial plane, the dominant amplitude is $\sim \mathcal{K}_o$. The spacetime around the rotating mass is described by the Kerr metric linearized in the angular momentum parameter. The determination of the relative acceleration between neighboring test masses amounts to calculating the tidal matrix with elements

$$\mathbf{K}_{ij}(\tau) = \mathbf{R}_{\mu\nu\rho\sigma}\, \lambda^{\mu}_{(0)}\, \lambda^{\nu}_{(i)}\, \lambda^{\rho}_{(0)}\, \lambda^{\sigma}_{(j)} \quad , \tag{4}$$

which are the "electric" components of the Riemann curvature tensor as measured by an observer following the center-of-mass of the system of test particles and carrying along it a parallel-transported system of orthonormal spatial directions ("gyroscopes"). Here τ is the proper time along the path, $\lambda^{\mu}_{(0)} = dx^{\mu}/d\tau$ denotes the vector tangent to the geodesic path of the center-of-mass and $\lambda^{\mu}_{(i)}$, i = 1,2,3, denote the spatial directions. Let us suppose that in the absence of rotation the center-of-mass describes a circular orbit around M, tilted at a small angle α . Furthermore, let $\hat{r}$, $\hat{t}$ and $\hat{n}$ be three orthogonal unit vectors ($\hat{n} = \hat{r} \times \hat{t}$) along the orbit denoting the radial, tangential and normal directions, respectively. The new unexpected terms in the (symmetric and tracefree) matrix K_{ij} are

$$\mathbf{K}_{\hat{r}\hat{n}} = \mathbf{K}_{\hat{n}\hat{r}} = 3\alpha \left(\frac{GM}{c^2 r}\right)^{-1} \mathcal{K}_0 \sin(\omega\tau + \eta_0) \quad , \tag{5}$$

where ω is the proper orbital frequency

$$\omega = \omega_0 \left(1 - 3\,\frac{GM}{c^2 r}\right)^{-1/2} \quad , \tag{6}$$

i.e., $2\pi/\omega$ is the orbital period according to the proper time of the orbiting observer, and η_o is a constant. The restriction of equation (5) to small angles of inclination may be removed by using the interesting work of Marck /6/, who has given an exact expression for the tidal matrix of the Kerr metric for an arbitrary timelike geodesic path. The result is that for arbitrary α, one must replace α in equation (5) by essentially sin α.

The amplitude of the new term in equation (5) is proportional to $\omega_o^2 \xi$, where

$$\xi = \frac{J}{M r^2 \omega_o} \tag{7}$$

is less than (or at most of order) unity, and ω_o^2 is of the order of the Newtonian expression for the relative tidal acceleration per unit separation of the test masses. It is interesting to note that ξ is independent of the speed of light, though equation (5) is a purely general relativistic effect: This off-diagonal tidal acceleration decreases with r as $r^{-7/2}$ and changes sign if the sense of rotation of the source is reversed. The independence of the new term from c indicates that it could be much larger than anticipated. Indeed, if α is not too small and $GM/c^2r \ll 1$, the new effect could be far greater than the original estimate given by equation (1). The reason for this may be traced back to the appearance of a small denominator, $\omega - \omega_o$, in the spatial tetrads. The existence of such a term is not in conflict with the results of Fokker /7/ and Schiff /8/, though the spatial tetrads describe the precession of gyroscopes along the orbit. This is because the new term contributes only a very slowly varying part to the spin vector of a gyroscope. The magnitude of this almost constant vector for small α is

$$\frac{3}{2}\alpha \frac{GJ}{c^2 r^3} \cdot \frac{\cos[(\omega-\omega_0)\tau + \eta_0]}{\omega-\omega_0} , \qquad (8)$$

which is essentially $\alpha\xi$, up to a phase factor.

The new gravitational effect of a rotating mass has the appearance of a resonance between the orbital frequencies ω and ω_0 . Thus it vaguely resembles the small divisor phenomenon familiar from the Newtonian treatment of the gravitational many-body problem. A classic example is provided by the Sun-Jupiter-Saturn system: The two frequencies of orbital motion around the Sun are commensurate, hence a small denominator occurs in the expression for the orbital perturbation of one planet due to the other. It must be emphasized that while a similarity with a known phenomenon in Newtonian mechanics exists, there is hardly a one-to-one correspondence. The fundamental reason for the existence of our new effect is the deviation of orbital frequency according to a local clock from that determined by an inertial clock.

To derive the astronomical consequences of the new effect, it is necessary to develop a method for the approximate treatment of the relativistic many-body problem. The methods already available /9-12/ are based on coordinate-dependent Newtonian concepts without much regard for the truly measurable quantities. It is possible, however, to cover the spacetime manifold with intersecting Fermi patches such that the application of the principle of equivalence to each patch would lead uniquely to the physically significant quantities. Consider, for example, the gravitational influence of the Sun on the Earth-Moon distance. This distance is much smaller than the Sun-Earth distance, hence a useful Fermi coordinate frame can be set up along the (approximately) geodesic path of the center-of-mass of the Earth-Moon system. The gravitational field in this frame is mainly due to the Earth and Moon, together with the perturbing influence of the tidal field of the Sun. Therefore, the influence of the Sun on the Earth-Moon distance consists of a dominant Newtonian tidal part of order $\sim (\omega_0/\Omega_0)^2 R_0$, together with small relativistic corrections. Here Ω_0 is the orbital frequency of the Moon around the Earth and R_0 is the Earth-Moon distance. The influence of the post-Newtonian spherical field of the Sun is, according to our analysis

$$\sim \left(\frac{GM}{c^2 r}\right)\left(\frac{\omega_0}{\Omega_0}\right)^2 R_0 \, , \tag{9}$$

which amounts only to a few centimeters. This is in contrast to the assertions in the literature /10,11,13/, which claim that the dominant perturbation in this case is $\frac{1}{4}(GM/c^2 r)\, R_0 \cong 100$ cm. This result, which is independent of how strongly the Earth-Moon system is bound, must be an artifact of the method of calculation. It cannot show up in the Earth-Moon observations.

The rotation of the Sun causes a harmonic variation in the Earth-Moon distance with an over-all amplitude of

$$\mathcal{A} = 3\,\alpha\,\beta\,\xi \left(\frac{\omega_o}{\Omega_o}\right)^2 R_o \quad , \tag{10}$$

where $\alpha \cong 7^o$ is the inclination of the ecliptic with respect to the equatorial plane of the Sun and $\beta \cong 5^o$ is the inclination of the Earth-Moon orbital plane with respect to the ecliptic. The magnitude of the solar angular momentum is

$$J_\odot = 1.7 \times 10^{48}\ j \quad \mathrm{gm\ cm^2\ sec^{-1}} \quad , \tag{11}$$

where j is expected to be of order unity /14,15/. Hence $\mathcal{A} = 1.4$ j meter, and the dominant frequencies in the orbital perturbation are $2\Omega_o - \omega_o - \tilde{\omega} \pm \omega$, $\omega_o + \tilde{\omega} \pm \omega$ and Ω_o , where $2\pi/\tilde{\omega} \cong 18.6$ yrs is the period of the retrograde motion of the line of nodes in the ecliptic. Thus the dominant periods are 18.6 years, $\sim 1/2$ year, 1 month, and $\sim 1/2$ month. This new one-meter effect is due to a small denominator given by $\omega - \omega_o$, as explained before. Could the effect disappear because of friction in the Earth-Moon system? The Moon moves away from the Earth by ~ 3 cm per year due to tidal friction. This implies a damping coefficient of $\Gamma \cong 2.4 \times 10^{-18}$ sec^{-1}, to be compared with $\omega - \omega_o \cong 3 \times 10^{-15}$ sec^{-1}. It appears, therefore, that the various effects in the Earth-Moon system will not alter our estimate (10) and that other small perturbations in the orbit superpose linearly with the new effect.

In the lunar laser ranging experiment /16/, the distance from the McDonald Observatory in Texas to retroreflectors on the

Moon has been measured with an accuracy of $\sim$ 10 cm. The net rms residual for the past decade corresponding to the difference between the observed and predicted ranges is, however, $\sim$ 40 cm. It appears, therefore, that this experiment is capable of testing our theoretical predictions by incorporating a new force given by equation (5) in the theoretical model for the Earth-Moon system, thereby determining the new parameter j from a least-squares fit to the data. The confirmation of our theoretical results by the data would provide a significant new test of Einstein's theory of gravitation and a direct measurement of the angular momentum of the Sun.

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