An essay on Gravitation

A proof of the cosmic censorship hypothesis

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Abstract.

It is shown that in a weakly asymptotically simple and empty space, according to classical general relativity, from non-singular initial data no strong curvature singularity can arise that is visible from infinity.

From theorems of Penrose, Hawking and Geroch it follows that singularities occur in a large class of physically reasonable space-times. The existence of singularities denotes the breakdown of our present theory of gravitation. A question comes to mind whether these uncontrollable situations can influence the state of space-time. Supposition that they cannot was put forward by Penrose (/1/) and it is known as the cosmic censor hypothesis CCH. The main difficulty with this hypothesis lies in the fact that there exists a number of exact solutions of the Einstein equations in which singularities are visible to observers. The problem is to find statements eliminating these cases.

One can hope to prove a formulation of CCH involving some assertion such as:

In a physically reasonable space-time a system which evolves from non-singular initial data and according to classical general relativity does not develop space-time singularities that are visible from infinity (cf. /2/).

Singularities that are visible from infinity are called naked.

One can justify the above statement as follows:

1. If there is a space-time singularity on the surface on which the initial data are set then one should not be surprised that

the singularity persists in the future of the initial surface and that it is visible from infinity if the initial surface is.

2. It is believed that as a result of the quantum evaporation process calculated by Hawking the black hole radiates away its mass leaving behind a naked singularity. Therefore it is better to restrict oneself to classical general relativity.

3. It is easy to create a nakedly singular space-time by simply removing a point of (a) space-time in the past of its infinity. Therefore one must restrict somehow the class of singularities under conside-ration.

I shall attempt to make the above formulation of CC mathematically precise. I shall confine myself to weakly asymptotically simple and empty (WASE) spaces (/3/ p.225). Such a space-time $(\mathcal{M}, \bar{g})$ can be conformally imbedded in a larger space-time $(\tilde{\mathcal{M}}, \tilde{g})$ as a manifold with boundary $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$, where the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ consists of two null surfaces $\mathcal{J}^+$ and $\mathcal{J}^-$ which represent future and past null infinity respectively. In these space-times there is an elegant statement of the CCH namely the future asymptotic predictability from a partial Cauchy surface $\mathcal{S}$ (/3/ p.310) which says that $\mathcal{J}^+$ is contained in the closure of $D^+(\mathcal{S})$ in the conformal manifold $\tilde{\mathcal{M}}$.

The following concept makes precise the idea of non-singular initial data:

Definition 1

A WASE space is partially future asymptotically predictable from a partial Cauchy surface $\mathcal{S}$ if the intersection of the closure of $D^+(\mathcal{S})$ in the conformal manifold $\tilde{\mathcal{M}}$ with $\mathcal{J}^+$ is not empty.

Throughout the rest of this essay I shall denote the intersection described in definition 1 by $\mathcal{J}_0^+$.

From the proof of the proposition 9.2.1 in /3/ one has the following lemma:

Lemma 1

Let $(\mathcal{M}, \bar{g})$ be partially future asymptotically predictable from $\mathcal{S}$ then for any compact two-surface $\mathcal{T}$ that intersects $J^+(\mathcal{S}) \cap I^-(\mathcal{J}_0^+)$ a null geodesic generator of $\dot{J}^+(\mathcal{T}, \tilde{\mathcal{M}})$ intersects $\mathcal{J}_0^+$.

For any two-surface $\mathcal{T}$ there are two families of the future-directed null geodesics orthogonal to it. If $\mathcal{T}$ intersects $J^+(\mathcal{S}) \cap I^-(\mathcal{J}_0^+)$ then any of the null geodesics that are members of that family which contains the generator that intersects $\mathcal{J}_0^+$ I shall call <u>outgoing</u>.

I shall use the language of TIPs and TIFs introduced by Geroch et al. /14/ to describe space-time singularities. From now on I assume that strong causality holds in space-time.

Definition 2

A subset Y of a WASE space such that Y is a TIP and $Y \subset I^-(p)$ for some point $p \in I^-(\mathcal{J}^+)$ is said to be a NSTIP.

It follows from the above definition and the definition of the WASE space that NSTIP cannot be of the form $I^-(p)$ where p is a point of $\partial\mathcal{M}$. Therefore NSTIP can be thought of as representing a singularity and, by definition 2, the one which is naked.

I shall now show that future asymptotic predictability is indeed equivalent to space-time being free of naked singularities arising from regular initial data.

Lemma 2

Let $(\mathcal{M}, \bar{g})$ be partially future asymptotically predictable from $\mathcal{S}$ then $(\mathcal{M}, \bar{g})$ is future asymptotically predictable from $\mathcal{S}$ iff there exists no subset X of $\mathcal{M}$ such that X is a NSTIP, $I^+(\mathcal{S}) \cap X \neq \emptyset$ and $I^+(\mathcal{S}) \cap X \subset I^+(\mathcal{S}) \cap I^-(\mathcal{J}_0^+)$.

Proof:

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If a NSTIP described above exists then by definition 2 there is a point p in $I^-(\mathcal{J}^+)$ such that $X \subset I^-(p)$. Thus there is a PIP that contains a TIP consequently by a theorem of PEnrose (/2/ section 12.3.2) the set $I^+(\mathcal{S}) \cap I^-(\mathcal{J}^+)$ cannot be globally hyperbolic and therefore by definition and /3/ prop.6.6.3 $(\mathcal{M}, \bar{g})$ cannot be future asymptotically predictable from $\mathcal{S}$.

$\Leftarrow$ Suppose that $(\mathcal{M}, \bar{g})$ is not future asymptotically predictable from $\mathcal{S}$. Let λ be a null generator of $\mathcal{J}^+$ that intersects $\mathcal{J}^+_0$ and let q be a point of λ in $\mathcal{J}^+_0$ where λ leaves $\mathcal{J}^+_0$. Such a point q exists since by definition 2 $\mathcal{J}^+_0$ is a closed subset of $\mathcal{J}^+$ and by hypothesis $\mathcal{J}^+_0 \neq \mathcal{J}^+$. Suppose that there is no set X in $I^-(q)$ such that X is a NSTIP, X intersects $I^+(\mathcal{S})$ and $X \cap I^+(\mathcal{S}) \subset I^+(\mathcal{S}) \cap I^-(\mathcal{J}^+_0)$. Since $I^-(q)$ and TIPs are open sets there is a neighbourhood $\mathcal{U}$ of q in the conformal manifold $\tilde{\mathcal{M}}$ such that no point $q \in \mathcal{U}$ contains in its chronological past a NSTIP that intersects $I^+(\mathcal{S})$. Consider any point r in $\lambda \cap \mathcal{U} \cap (\mathcal{J}^+ - \mathcal{J}^+_0)$. Since $r \notin \mathcal{J}^+_0$ the set $I^-(q) \cap I^+(\mathcal{S})$ is not globally hyperbolic. Thus by /2/ section 12.3.2 there exists a point s in $I^-(q) \cap I^+(\mathcal{S})$ such that $I^-(s)$ contains a TIP. Clearly this TIP must intersect $I^+(\mathcal{S})$. By definition 2 such a TIP is a NSTIP and obviously it is in the past of q. This is a contradiction.

I shall now come to the problem of singular space-time which are to be considered physically reasonable. There are well-known examples of space-times, constructed by Yodzis et al. (/5/, /6/) , in which naked singularities do develop from non-singular initial data.

These singularities arise as a result of intersection of spherically symmetric dust shells. There is a common feeling among researchers that there is something unphysical about shell-crossing. The problem is to find exactly what (/2/, /7/). The key observation seems to come from Seifert (/8/ see also Tipler et al. /9/), namely, that observers falling into the shell-crooosing singularity experience only final tidal stresses. This is in a sharp contrast with the properties of the Schwarzshild singularity where observers are crushed to zero volume by infinite tidal stresses (/10/ p.860). The singularities of the type occurring in the Schwarzschild space-time are called strong curvature singularities. Their idea was introduced by Ellis and Schmidt (/11/) and they were defined precisely by Tipler (/12/). Here I shall give my own definition based on the definition of Tipler et al. (/9/).

Definition 3 (/13/)

Let the set Z be a TIP such that any causal geodesic γ for which $I^-(\gamma) = Z$ is future-incomplete. Let μ be a 3-form on the normal space to the tangent vector of γ determined by three independent vorticity-free Jacobi fields Z_1, Z_2, Z_3 along γ (if γ is null, is defined as a 2-form i.e. $\mu = Z_1 \wedge Z_2 \wedge Z_3$. Z is said to be a SSTIP if: $(\forall \mu)(\forall p \in \gamma)(\forall \varepsilon > 0, \varepsilon \in R)(\exists q(\mu, p \varepsilon) \in \gamma \cap J^{+}(p) : \|\mu(q)\| < \varepsilon)$

i.e. if for all μ and for any point p on γ and any real number $\varepsilon > 0$ there exists a point q (depending on μ, p ε) on γ in $J^{+}(p)$ such that $\|\mu\| < \varepsilon$ at q.

SSTIPs are said to represent strong curvature singularities.

Thus according to the above definition any geodesic observer approaching the strong curvature singularity is crushed to zero volume at or before or at the singularity. Strong curvature singularities singularities are well defined geometrically and ,in my opinion, they make precise the intuitive feeling of what a physically important singularity should be. I have put forward a hypothesis that "singularities in all reasonable physical cases are of strong curvature type" (/14/ p.72) .A similar idea was also given by Tipler et al. (/9/), they suggest that "in any physically realistic space-time, all incomplete causal geodesics terminate in strong curvature singularities".

I am now in position to give a precise statement of the cosmic censorship hypothesis that will be possible to prove.

Theorem 1

Let $(\mathcal{M}, \bar{g})$ be partially future asymptotically predictable from a partial Cauchy surface $\mathcal{S}$ and suppose that the following conditions hold in $(\mathcal{M}, \bar{g})$:

a. $(\mathcal{M}, \bar{g})$ is strongly causal ,

b. $R_{ab}k^a k^b \geqslant 0$ for every null vector k^a ,

c. each NSTIP is a SSTIP ,

d. for any subset V of $\mathcal{M}$ which is an IP such that $V \subset I^-(\mathcal{J}_0^+)$ and the intersection $I^+(\mathcal{S}) \cap V$ is not empty there exists a null geodesic generator λ of $\dot{V}$ in $I^+(\mathcal{S})$ such that λ is an outgoing null geodesic

then $(\mathcal{M}, \bar{g})$ is future asymptotically predictable from $\mathcal{S}$.

Proof:

Suppose that $(\mathcal{M}, \bar{g})$ is not future asymptotically predictable from $\mathcal{S}$ then $\mathcal{J}_0^+ \neq \mathcal{J}^+$ consequently by lemma 2 there exists a NSTIP X in $I^-(\mathcal{J}_0^+)$ such that $I^+(\mathcal{S}) \cap X \neq \phi$ .Let γ be an outgoing null geodesic generator in $\dot{X}$ .By /4/ theorem 2.3 the set $Y = I^-(\gamma)$ is a TIP.Clearly Y must be a NSTIP and therefore ,by condition c. a SSTIP.By definition 3 the expansion $\hat{\theta}$ of a congruence of the null geodesics containing γ and generating $\dot{Y}$ must become negative. One can now use the well-known argument of the proof of lemma 9.2.2

in /3/ and arrive at a contradiction. In this last argument condition b. is necessary.

Condition a. above is an obvious physical requirement. Condition b. means that one restricts oneself to classical relativity. Condition c. is a precise statement of the hypothesis I have put forward above. Condition d. is an additional demand which insures that the naked singularity is not on the initial surface $\mathcal{S}$ (see fig. 1).

On figure 2 I have drawn a hypothetical space-time in which a naked singularity arises from non-singular initial data. What the above theorem shows is that this singularity cannot be of strong curvature type.

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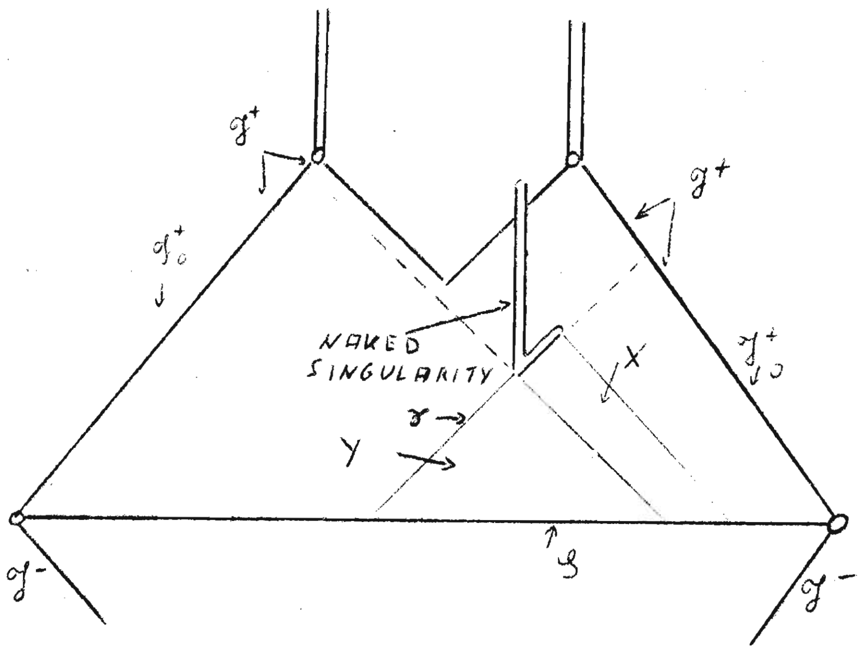

Figure 2

This figure serves as a illustration for the proof of the theorem 1.

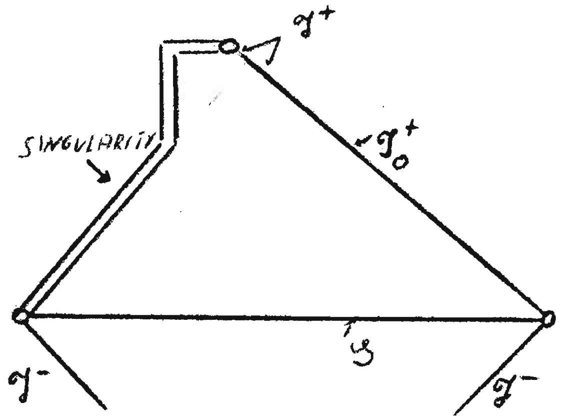

Figure 1

A hypothetical space-time that is partially future asymptotically predictable from $\mathcal{S}$ but in which there is a naked singularity in every neighbourhood of $\mathcal{S}$ .Some condition such as condition d. in the theorem 1 must be imposed to eliminate these situations.