

NO MORE SPACETIME SINGULARITIES? ¹

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We discuss the possibility that the issue of spacetime singularities in general relativity is solved by their stringy extensions.

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General relativity is a classical theory of gravitation and spacetime. Perhaps its most spectacular success is its application to the universe as a whole and the related description of big-bang cosmology from an era of about 10^{-35} seconds until the present. Nonetheless, there are two major difficulties with the theory. The first is a problem afflicting any classical theory, namely, whether it can be derived as the classical limit of some consistent quantum theory. The second difficulty is that, even as a classical theory, general relativity is deficient as a theory of spacetime because it predicts the existence of singularities.

The singularity theorems of Hawking and Penrose [1, 2] assert that a spacetime is geodesically incomplete provided that there is a reasonable sense of causality, that the generic condition holds, that there is either a trapped surface or a general cosmological expansion, and that the timelike convergence condition holds. The latter is the requirement that

$$R_{\mu\nu}k^\mu k^\nu \geq 0 \tag{1}$$

is satisfied for arbitrary timelike vectors k^μ . In particular, the theorems imply that the spacetimes associated with both gravitational collapse and cosmological expansion are geodesically incomplete. In the standard examples, the Schwarzschild solution or the Friedmann-Robertson-Walker universes, this incompleteness arises as a consequence of the infinite curvature encountered along some spacelike surface. However, the singularity theorems guarantee that this is a generic problem rather than some difficulty arising from over-restrictive assumptions in the derivation of these particular solutions.

The physical meaning of geodesic incompleteness is that a geodesic terminates at a finite proper time in the past, in the future, or both. An observer moving along such a geodesic would reach the boundary of spacetime. The problems posed by this apocalyptic prediction of general relativity are insurmountable, at least within the context of classical physics.

The physical reason for the singularity theorems is that gravitation is universally attractive. Consider a congruence of timelike geodesics parametrized by proper time s along the curves $x^\mu(s)$ with tangent vector $k^\mu = dx^\mu/ds$. The volume expansion

$\theta = \nabla^\mu k_\mu$ of the congruence satisfies the Raychaudhuri equation [2]

$$\frac{d\theta}{ds} = -R_{\mu\nu}k^\mu k^\nu + \dots \quad . \quad (2)$$

The right-hand side of this equation consists of effectively negative quantities, with the exception of the first term. The first term is also negative provided that the energy-momentum tensor of matter obeys certain plausible conditions. Using the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \quad (3)$$

we see that Eq. (1) holds if

$$(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})k^\mu k^\nu \geq 0 \quad . \quad (4)$$

This is the strong energy condition, satisfied for most forms of macroscopic classical matter.

One might hope that an underlying quantum-mechanical description of gravitation would resolve the issue of classical singularities. The only candidate theory available at present appears to be string theory, which is based on the idea that the fundamental structure of an elementary object is a two-dimensional world sheet (rather than the one-dimensional world line of point particles) together with the principle of conformal invariance. If the string world sheet Σ has a metric γ_{ab} , carries coordinates ξ^a , $a = 1, 2$, and has location in d -dimensional spacetime given by $X^\mu(\xi^a)$, then its dynamics is determined by the action [3]

$$I = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\xi \sqrt{\gamma} \left(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + \frac{1}{2}\alpha' T - \frac{1}{4}\alpha' R^{(2)}\phi \right) \quad . \quad (5)$$

Here, $R^{(2)}$ is the Ricci curvature scalar formed from the metric γ_{ab} and $g_{\mu\nu}$ is the background metric of a curved spacetime, coming from the combination of the string graviton $h_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$. The tachyon T and the dilaton ϕ are fields for the lightest modes of the string. The only dimensionful parameter of the theory is α' , which sets the scale at which stringy effects become important. To date, the theory remains incomplete because it is not understood how spacetime emerges from string theory or why spacetime itself is related to the excitations of the string

that describe gravitons. Despite this lacuna, there is evidence that the theory is internally consistent with an identification between these two apparently unrelated concepts.

Although I has conformal invariance to lowest order in α' , this symmetry will be violated quantum-mechanically unless certain conditions hold. For the action (5), these conditions are:

$$R_{\mu\nu} = \nabla_\mu \nabla_\nu \phi + \nabla_\mu T \nabla_\nu T + \alpha' (R_{\mu\lambda\rho\sigma} R_\nu^{\lambda\rho\sigma}) + O(\alpha'^2) \quad , \quad (6)$$

$$\square T + (\nabla_\mu \phi)(\nabla^\mu T) + 2T = O(\alpha') \quad , \quad (7)$$

$$\frac{2}{3}(26 - d) + \alpha' [R - (\nabla\phi)^2 - 2\square\phi - (\nabla T)^2 - 2T^2] + O(\alpha'^2) = 0 \quad . \quad (8)$$

These conditions are the string replacements of the Einstein equations, into which they degenerate in the limit $\alpha' \rightarrow 0$.

It is interesting to note that the stringy modifications in Eq. (6) have no definite sign, and therefore there is no reason to believe that the timelike convergence condition (1) is satisfied in string extensions of general relativity. Nonetheless, the conditions (6) - (8) cannot be directly used to probe the existence of spacetime singularities. There are two reasons for this. The first is a practical issue. If singularities are to be avoided, it will be because the stringy corrections to the Einstein equations become large (since the lowest-order equations do obey the timelike convergence condition). However, if the first correction is large because the curvature is large on the string scale, all higher-order terms are large too, and Eqs. (6)- (8) become prohibitively difficult to solve beyond low orders. Second, since the treatment is inherently perturbative in powers of α' , the starting point must be a spacetime that is itself singular. However, no amount of perturbation will ever remove the singularity, and so any attempts to resolve the issues based on these equations will necessarily be stymied.

Let us therefore attempt to find backgrounds that result from exactly conformally invariant string theories. We mostly restrict our attention to spacetime dimension $d = 2$ to simplify matters. The first background that we consider is the so-called linear dilaton background [4]. The spacetime is just two-dimensional Minkowski spacetime with $T = 0$ and $\phi = \phi_0 + \lambda r$, where $\lambda = 4/\sqrt{\alpha'}$. It satisfies Eqs. (6)- (8) to lowest

order, and all higher-order corrections vanish by virtue of the fact that spacetime is flat and that $\nabla_\mu\phi$ is a Killing vector. It is satisfying to discover that Minkowski spacetime is an acceptable background.

The second example is provided by a conformal field theory that emerges from the GKO prescription [5] for the coset $SU(1,1)_{-k}/U(1)$. The exact spacetime background corresponding to this theory is given by [6]

$$ds^2 = -dt^2(\coth^2 r - \frac{2}{k})^{-1} + dr^2 \quad (9)$$

and

$$\phi = \phi_0 + \ln \left[\frac{\sinh 2r}{(\coth^2 r - 2/k)^{1/2}} \right] \quad (10)$$

If one takes the limit $k \rightarrow \infty$, then this spacetime obeys Eqs. (6)-(8) with $\alpha' = 0$ [7]. However, exact conformal invariance is achieved only for the case $k = 9/4$.

The spacetime given by Eqs. (9)-(10) is a black-hole spacetime. It is asymptotically flat as $r \rightarrow \infty$, and there is a Killing horizon at $r = 0$. The metric (10) can be put into ‘canonical’ Schwarzschild form by the coordinate transformation $x = \cosh 2r$, so that [8]

$$ds^2 = -\frac{x-1}{x-x_c} dt^2 + \frac{1}{(x-1)(x+1)} dx^2 \quad , \quad (11)$$

where

$$x_c = \frac{2+k}{2-k} \quad (12)$$

For $k > 2$, $x_c < -1$. The coordinate singularity at $r = 0$ or $x = 1$ is now of a standard form and can be removed by the usual Kruskal construction. A calculation of the curvature shows that R is finite except for $x = x_c$. However, this is not part of the original spacetime $r \geq 0$, since to get to x_c from $x \geq -1$ would require travel through a region of positive euclidean signature.

In fact, the entire region $x > -1$ has a geodesically complete maximal analytic extension shown in Figure 1 [9]. The regions I are asymptotically flat regions with $x > 1$, and the regions II are regions with $-1 < x < 1$ interior to the horizon. The surface $x = -1$ is just a coordinate singularity and is the surface of time-reflection symmetry indicated by the dotted lines. Our conclusion therefore is that in string

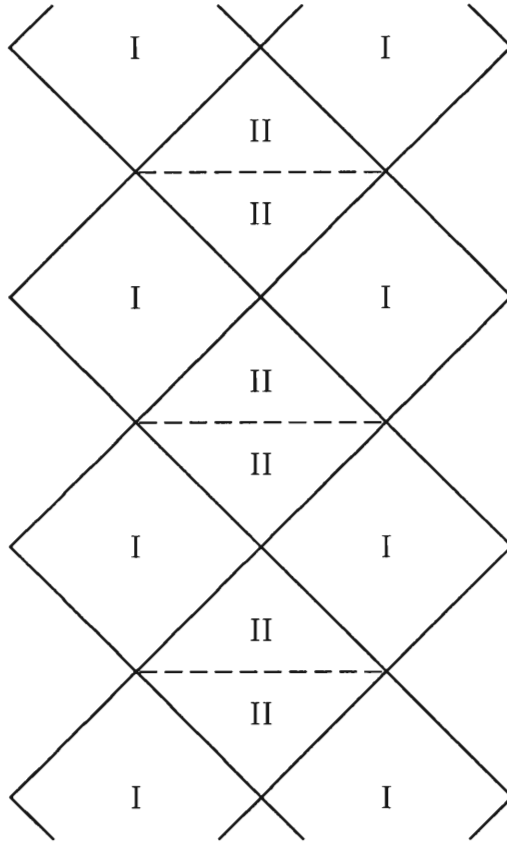


Fig. 1. Penrose diagram for the extended metric (12) in the region $x \geq -1$.

theory, at least in this special case, black holes do not have any associated spacetime singularity.

Note that the absence of a singularity here may help resolve the Hawking paradox [10]. This is the problem that quantum information such as phase structure is lost whenever matter crosses a spacetime boundary at a singularity. If there are no singularities in string theory, as is suggested by the above example, information will be globally conserved. Any information that appears lost in one asymptotically flat region reappears instead in another, which becomes a white-hole spacetime.

For the case $d > 2$, the possibility of performing conformal transformations without affecting string physics can offer additional freedom in reinterpreting apparently singular solutions. We can consider, for example, a class of Friedman-Robertson-Walker universes with line element

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + \dots dx_{d-1}^2) \quad . \quad (13)$$

The lowest-order solutions of this form to Eqs. (6) -(8) are known [11], and we can use them to examine our hypothesis. For definiteness, consider the solution

$$a(t) = a_0 t^{1/\lambda} \quad , \quad \phi(t) = \phi_0 + (1 - \lambda) \ln(t/\sqrt{\alpha'}) \quad . \quad (14)$$

Here, $\lambda = (d-1)^{1/2}$, and a_0 and ϕ_0 are arbitrary constants of integration. This spacetime is singular when $a(t) \rightarrow 0$. The universe then has zero size, corresponding to the big bang. However, string theory is invariant under spacetime conformal transformations that may well convert an apparently singular spacetime into a nonsingular one, without changing the string physics. In the present example with $d > 2$, a conformal rescaling of the metric by a factor $e^{2\phi/\lambda(\lambda-1)}$ converts the spacetime into a flat and hence nonsingular one.

The results we have presented here suggest that string-metric singularities are harmless, unlike apparently similar solutions of general relativity containing one or more spacetime boundaries. If this is generically true, the issue of singularities in general relativity is resolved by string theory.

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