Black Holes and the Stability of Gravitation

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Abstract

The existence of black holes in general relativity provides an effective cut-off to the negative gravitational potential. This results in a fundamental upper limit on the amount of energy that can be radiated away by any isolated system.

* Supported by National Science Foundation Grant PHY80-19754

Is it possible to extract an infinite amount of energy from a finite system? Can there exist objects that have a total mass which is negative? Such objects would generate a gravitational repulsion rather than an attraction. It is perhaps rather surprising that these questions have been answered only quite recently. To understand why the answers to these questions are not "obvious", we begin by considering the Newtonian theory of gravitation.

In Newtonian theory, one can obtain unlimited amounts of energy from an isolated system. Consider, for example, a system consisting of two small masses in orbit around each other. Since the system is gravitationally bound, the total energy is negative. Since the Newtonian potential is unbounded below, we can make the total energy as large and negative as we wish by making the orbit smaller and smaller. In Newtonian gravity, we can even construct point masses, so the entire system could in principle be made arbitrarily small, leading to an infinitely negative energy.

There is, of course, something rather wrong with this picture. The laws of thermodynamics rule out the possibility of extracting infinite amounts of energy from finite systems. Moreover, if we start to include the effects of special relativity, the matter gets rather worse. If we just naively add in the rest mass energy of the two small masses, it seems that the total relativistic energy of such a system could be negative. Special relativity tells us that energy is equivalent to inertial mass, and the Principle of Equivalence in turn tells us that inertial and gravitational mass are identical. Thus, if such systems existed we would have gravitational repulsion or "anti-gravity". However, such an isolated system would have a total energy seen by a distant observer of $E = E_0(1-v^2)^{-\frac{1}{2}}$ with $E_0 < 0$, where v is the system's velocity relative to the observer. (We use units in which $G = c = 1$ throughout.) Physical systems in equilibrium will tend to evolve to their lowest energy state, so our hypothetical system will spontaneously accelerate to the speed of light.

Einstein's General Theory of Relativity is supposed to cure the inconsistency between Newtonian theory and special relativity. Indeed, here the situation looks more hopeful since it seems that if we make our two body system sufficiently small, we will create a black hole which should act so as to limit the amount of energy which can be extracted, and hence ensure that negative energy systems do not exist. The purpose of this essay is to show that this is indeed the case.

One can define the total energy of an isolated self-gravitating system which includes contributions from both the gravitational field and matter fields. This total energy is measured at spacelike infinity and is referred to as the Arnowitt-Deser-Misner Energy[1], E_{ADM} . One can also define the energy in an isolated system "left over" after radiation has been emitted. This energy is measured at (future) null infinity and is called the Bondi energy E_B [2] . Unlike E_{ADM} , the Bondi energy is a function of retarded time u . Of course, the energy radiated away between $u = -\infty$ and $u = u_0$ is the difference $E_{ADM} - E_B(u_0)$. It was shown twenty years ago that $E_B(u)$ is in fact a decreasing function of u . That is, radiation always carries away positive energy. The question we wish to consider is whether $E_B(u) \geq 0$ for all u . If not, then E_B can presumably decrease indefinitely and an infinite amount of energy may be lost from the system.

The question of whether there exist systems with negative total energy in general relativity would appear to be very difficult to answer. The total energy of a system is defined by the asymptotic behavior of its gravitational field. This field is related to the matter sources in an extremely complicated way via Einstein's equations.

Preliminary results on the positivity of energy were obtained by considering simplifying assumptions, for example weak fields or special symmetries[3]. The first complete proof that $E_{ADM} \geqslant 0$ was given by Schoen and Yau in 1979[4]. Their proof was very geometrical and involved extremal surfaces. About a year later, an alternative proof of this result was given by Witten[5].

In this proof, one solves a linear elliptic differential equation for a spinor field on a complete spacelike hypersurface Σ which stretches out to spacelike infinity. One can show that the ADM mass can be expressed in terms of an integral involving this spinor field on $\partial\Sigma$ which is at spacelike infinity. One then converts this into an integral over Σ of a quantity which is positive definite, provided that the energy momentum tensor of matter is positive (that is, obeys the dominant energy condition[6]). This shows that $E_{ADM} \geqslant 0$. When first presented, this proof seemed somewhat miraculous. However, recent work[7,8] has led to a better understanding of the result, and also to a generalization which shows that $E_B \geqslant 0$, even in the presence of black holes[9]. It is now possible to present the argument in a logically motivated fashion. (These results have also been proved by Schoen and Yau using their techniques[10].)

The Bondi 4-momentum P_a^B of an isolated system may be conveniently defined by a generalization of the Komar integral for the conserved quantity associated with a physical symmetry. Let K^a be a conserved null vector field whose restriction to future null infinity $\overset{\circ}{K}{}^a$ is an asymptotic translation. Then $\overset{\circ}{K}{}^a P^B_a$ is defined to be the integral of $\frac{1}{8\pi}\,\varepsilon^{abcd}\,\nabla_{[a}K_{b]}$ over an asymptotic two-sphere S near null infinity. If we convert this to a volume integral over a complete spacelike hypersurface Σ whose boundary is S , the integrand contains two derivatives and the K^a . Since we want the integrand to be positive, we would like to express it in the form $(\nabla\alpha)^2$ for some α . Thus α must be the "square root" of K^a i.e. α must be spinorial.

Let $K^a = \alpha^A \bar{\alpha}^{A'}$ where α^A is a two component spinor. With no restrictions on α^A (other than $\alpha^A \to \overset{\circ}{\alpha}{}^A$ constant) the volume integral becomes

$$\overset{\circ}{K}{}^a P^B_a = \frac{1}{8\pi}\int_\Sigma \{\nabla_a(\nabla_m K^m) - R_{am}K^m - \bar{\alpha}_{A'}\nabla^2\alpha_A - \alpha_A\nabla^2\bar{\alpha}_{A'} -2(\nabla_m\alpha_A)(\nabla^m\bar{\alpha}_{A'})\}d\Sigma^{AA'} \tag{1}$$

Since we have an inequality on G_{ab}-- via Einstein's equation and the dominant energy condition -- and not R_{ab} , we wish to choose α^A such that $\nabla^2\alpha_A = -\frac{1}{4}R\alpha_A$. Furthermore since the first term on the right has no definite sign we would like $\nabla_a K^a = 0$. The simplest way to achieve these two conditions is to require that α^A satisfies the Weyl neutrino equation: $\nabla_{AA'}\alpha^A = 0$. Equation (1) then becomes

$$\overset{\circ}{K}{}^a P^B_a = \int_\Sigma \{T_{ab}K^b - \frac{1}{4\pi}\nabla_m\alpha_A\nabla^m\bar{\alpha}_{A'}\}\, d\Sigma^{AA'} \tag{2}$$

The integrand is not yet positive because the second term can have either sign. However there is still some freedom left in the choice of α^A . We now choose α^A such that $\hat{t}^m \nabla_m \alpha^A = 0$ on Σ where $\hat{t}^a$ is a future directed timelike vector which asymptotically approaches a time translation. (This is essentially a choice of initial data for the Weyl equation. One can show that there exist initial data which satisfy this condition and asymptotically approach a constant.) With this condition on α^A , the second term in equation (2) cannot be negative.

This proves that P_a^B is a future-directed timelike or null vector. Using similar arguments one can show[8,11] that $P_a{}^B$ is in fact strictly timelike unless the spacetime is flat in a neighbourhood of Σ in which case $P_a^B = 0$. This proof can be extended to deal with the case where black holes are present. By setting a boundary condition on the outermost trapped surface, one can show that the total mass must still be positive. It is tempting to try and give a physical interpretation to each of the terms in equation (2). On the left we have the "total energy remaining in the system". The first term on the right is clearly the contribution to this energy from the matter fields, and therefore the second term must be the contribution from the gravitational field. However, one cannot regard $\nabla_m \alpha_A \nabla^m \overline{\alpha}_{A'}$ as a measure of the gravitational energy density because it is not a local function of the geometry. The equation that the initial data must satisfy on Σ is an elliptic equation and hence a small change in the geometry near one point affects the solution everywhere.

Thus, the gravitational energy cannot become negative, and there is a fundamental upper limit on the amount of energy that can be extracted from a given physical system.

The key to these proofs is the observation that one can essentially replace Einstein's equations by a <u>linear</u> equation. This is truly a remarkable fact, and one whose implications have not yet been fully understood. Why is there such a simple relationship in general relativity? There are indications that this question is related to the question of why general relativity admits a supersymmetric extension, supergravity. Do there exist linear equations capturing other aspects of Einstein's equation? Perhaps the ideas discussed here will lead to further relations between asymptotic and local quantities, for example,angular momentum or higher multipole moments.

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