

Gravitation, the Quantum, and Bohr's Correspondence Principle

Shahar Hod

The Racah Institute for Physics, The Hebrew University, Jerusalem 91904, Israel

(March 1, 1999)

Abstract

The black hole combines in some sense both the “hydrogen atom” and the “black-body radiation” problems of quantum gravity. This analogy suggests that black-hole quantization may be the key to a quantum theory of gravity. During the last twenty-five years evidence has been mounting that black-hole surface area is indeed *quantized*, with *uniformly* spaced area eigenvalues. There is, however, no general agreement on the *spacing* of the levels. In this essay we use Bohr's correspondence principle to provide this missing link. We conclude that the fundamental area unit is $4\hbar \ln 3$. This is the unique spacing consistent both with the area-entropy *thermodynamic* relation for black holes, with Boltzmann-Einstein formula in *statistical physics* and with *Bohr's correspondence principle*.

Everything in our past experience in physics tells us that general relativity and quantum theory must be approximations, special limits of a single, universal theory. However, despite the flurry of research, which dates back to the 1930s, we still lack a complete theory of quantum gravity. It is believed that black holes may play a major role in our attempts to shed some light on the nature of a quantum theory of gravity (such as the role played by atoms in the early development of quantum mechanics).

The quantization of black holes was proposed long ago in the pioneering work of Bekenstein [1]. The idea was based on the remarkable observation that the horizon area of nonextremal black holes behaves as a classical *adiabatic invariant*. In the spirit of Ehrenfest

principle, any classical adiabatic invariant corresponds to a quantum entity with a *discrete* spectrum, Bekenstein conjectured that the horizon area of a quantum black hole should have a discrete eigenvalue spectrum.

To elucidate the *spacing* of the area levels it is instructive to use a semiclassical version of Christodoulou's reversible processes. Christodoulou [2] showed that the assimilation of a neutral (*point*) particle by a (nonextremal) black hole is reversible if it is injected at the *horizon* from a radial *turning point* of its motion. In this case the black-hole surface area is left unchanged and the changes in the other black-hole parameters (mass, charge, and angular momentum) can be undone by another suitable (reversible) process. (This result was later generalized by Christodoulou and Ruffini for charged point particles [3]).

However, in a *quantum* theory the particle cannot be both at the horizon and at a turning point of its motion; this contradicts the *Heisenberg quantum uncertainty principle*. As a concession to a quantum theory Bekenstein [4] ascribes to the particle a *finite* effective proper radius b . This implies that the capture process (of a neutral particle) involves an unavoidable increase $(\Delta A)_{\min}$ in the horizon area [4]:

$$(\Delta A)_{\min} = 8\pi(\mu^2 + P^2)^{1/2}b , \quad (1)$$

where μ and P are the rest mass and physical radial momentum (in an orthonormal tetrad) of the particle, respectively. In the classical case the limit $b \rightarrow 0$ recovers Christodoulou's result $(\Delta A)_{\min} = 0$ for a reversible process. However, a *quantum* particle is subjected to a quantum uncertainty – the particle's center of mass cannot be placed at the horizon with accuracy better than the radial position uncertainty $\hbar/(2\delta P)$. This yields a lower bound on the increase in the black-hole surface area due to the assimilation of a (neutral) test particle

$$(\Delta A)_{\min} = 4\pi l_p^2 , \quad (2)$$

where $l_p = \left(\frac{G}{c^3}\right)^{1/2} \hbar^{1/2}$ is the Planck length (we use gravitational units in which $G = c = 1$). Thus, for nonextremal black holes there is a *universal* (i.e., independent of the black-hole parameters) minimum area increase as soon as one introduces quantum nuances to the problem.

The universal lower bound Eq. (2) derived by Bekenstein is valid only for *neutral* particles [4]. Expression (1) can be generalized for a *charged* particle of rest mass μ and charge e . Here we obtain

$$(\Delta A)_{min} = \begin{cases} 4\pi[2(\mu^2 + P^2)^{1/2}b - e\Xi_+b^2] & , \quad b < b^* , \\ 4\pi(\mu^2 + P^2)/e\Xi_+ & , \quad b \geq b^* , \end{cases} \quad (3)$$

where Ξ_+ is the black-hole electric field (we assume that $e\Xi_+ > 0$) and $b^* \equiv (\mu^2 + P^2)^{1/2}/e\Xi_+$. Evidently, the increase in black-hole surface area can be *minimized* by maximizing the black-hole electric field. Is there a physical mechanism which can prevents us from making expression (3) as small as we wish ? The answer is “yes” ! *Vacuum polarization* effects set an upper bound to the strength of the black-hole electric field; the critical electric field Ξ_c for pair-production of particles with rest mass μ and charge e is $\Xi_c = \pi\mu^2/e\hbar$ [5]. Therefore, the minimal black-hole area increase is given by

$$(\Delta A)_{min} = 4l_p^2 . \quad (4)$$

Remarkably, this lower bound is *independent* of the black-hole parameters.

The underling physics which excludes a completely reversible process (for neutral particles) is the *Heisenberg quantum uncertainty principle* [4]. However, for *charged* particles it must be supplemented by another physical mechanism – a *Schwinger discharge* of the black hole (vacuum polarization effects). Without this physical process one could have reached the reversible limit. It seems that nature has “conspired” to prevent this.

It is remarkable that the lower bound found for charged particles is of the same order of magnitude as the one given by Bekenstein for neutral particles, even though they emerge from *different* physical mechanisms. The *universality* of the fundamental lower bound (i.e., its independence on the black-hole parameters) is clearly a strong evidence in favor of a *uniformly* spaced area spectrum for quantum black holes. Hence, one concludes that the quantization condition of the black-hole surface area should be of the form

$$A_n = \gamma l_p^2 \cdot n \quad ; \quad n = 1, 2, \dots , \quad (5)$$

where γ is a dimensionless constant.

It should be recognized that the precise values of the universal lower bounds Eqs. (2) and (4) can be challenged. This is a direct consequence of the *inherent fuzziness* of the uncertainty relation. Nevertheless, it should be clear that the fundamental lower bound must be of the same order of magnitude as the one given by Eq. (4); i.e., we must have $\gamma = O(4)$. The small uncertainty in the value of γ is the price we must pay for not giving our problem a full quantum treatment. In fact, the above analyses are analogous to the well known semiclassical derivation of a lower bound to the ground state energy of the hydrogen atom (calculated by using Heisenberg's uncertainty principle, *without* solving explicitly the Schrödinger wave equation). The analogy with usual quantum physics suggests the next step – a *wave* analysis of black-hole perturbations.

The evolution of small perturbations of a black hole are governed by a one-dimensional Schrödinger-like wave equation (assuming a time dependence of the form $e^{-i\omega t}$) [6]. Furthermore, it was noted that, at late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [7]. To describe these free oscillations of the black hole the notion of quasinormal modes was introduced [8]. The quasinormal mode frequencies (ringing frequencies) are characteristic of the black hole itself.

It turns out that there exist an infinite number of quasinormal modes for $n = 0, 1, 2, \dots$ characterizing oscillations with decreasing relaxation times (increasing imaginary part) [9]. On the other hand, the real part of the frequency approaches a *constant* value as n is increased.

Our analysis is based on *Bohr's correspondence principle* (1923): “transition frequencies at large quantum numbers should equal classical oscillation frequencies”. Hence, we are interested in the asymptotic behavior (i.e., the $n \rightarrow \infty$ limit) of the ringing frequencies. These are the highly damped black-hole oscillations frequencies, which are compatible with the statement (see, for example, [10]) “quantum transitions do not take time” (let $w = w_R - iw_I$, then $\tau \equiv w_I^{-1}$ is the effective relaxation time for the black hole to return to a quiescent state. Hence, the relaxation time τ is arbitrarily small as $n \rightarrow \infty$).

Nollert [11] found that the asymptotic behavior of the ringing frequencies of a Schwarzschild black hole is given by

$$Mw_n = 0.0437123 - \frac{i}{4} \left(n + \frac{1}{2} \right) + O \left[(n+1)^{-1/2} \right] . \quad (6)$$

It is important to note that the asymptotic limit is *independent* of the multipole index l of the perturbation field. This is a crucial feature, which is consistent with the interpretation of the highly damped ringing frequencies (in the $n \gg 1$ limit) as being characteristics of the *black hole* itself. The asymptotic behavior Eq. (6) was later verified by Andersson [12] using an independent analysis.

We note that the numerical limit $Re(Mw_n) \rightarrow 0.0437123$ (as $n \rightarrow \infty$) agrees (to the available data given in [11]) with the expression $\ln 3/(8\pi)$. This identification is supported by thermodynamic and statistical physics arguments discussed below. Using the relations $A = 16\pi M^2$ and $dM = E = \hbar\omega$ one finds $\Delta A = 4l_p^2 \ln 3$. Thus, we conclude that the dimensionless constant γ appearing in Eq. (5) is $\gamma = 4 \ln 3$ and the area spectrum for a quantum black hole is given by

$$A_n = 4l_p^2 \ln 3 \cdot n \quad ; \quad n = 1, 2, \dots . \quad (7)$$

This result is remarkable from a *statistical physics* point of view ! The semiclassical versions of Christodoulou's reversible processes, which naturally lead to the conjectured area spectrum Eq. (5), are at the level of mechanics, not statistical physics. In other words, these arguments did not rely in any way on the well known thermodynamic relation between black-hole surface area and entropy. In the spirit of Boltzmann-Einstein formula in statistical physics, Mukhanov and Bekenstein [13,10] relate $g_n \equiv \exp[S_{BH}(n)]$ to the number of microstates of the black hole that correspond to a particular external macrostate (S_{BH} being the black-hole entropy). Namely, g_n is the degeneracy of the n th area eigenvalue. The accepted thermodynamic relation between black-hole surface area and entropy [4] can be met with the requirement that g_n has to be an integer for every n only when

$$\gamma = 4 \ln k \quad ; \quad k = 2, 3, \dots . \quad (8)$$

Thus, statistical physics arguments force the dimensionless constant γ in Eq. (5) to be of the form Eq. (8). Still, a specific value of k requires further input, which was not available so far. The correspondence principle provides a first independent derivation of the value of k . It should be mentioned that following the pioneering work of Bekenstein [1] a number of independent calculations (most of them in the last few years) have recovered the uniformly spaced area spectrum Eq. (5) [14]. However, there is no general agreement on the spacing of the levels. Moreover, *non* of these calculations is compatible with the relation $\gamma = 4 \ln k$, which is a direct consequence of the accepted thermodynamic relation between black-hole surface area and entropy.

The fundamental area spacing $4l_p^2 \ln 3$ is the *unique* value consistent both with the area-entropy *thermodynamic* relation, with *statistical physics* arguments (namely, with the Boltzmann-Einstein formula), and with *Bohr's correspondence principle*.

ACKNOWLEDGMENTS

It is a pleasure to thank Jacob D. Bekenstein and Tsvi Piran for stimulating discussions. This research was supported by a grant from the Israel Science Foundation.

REFERENCES

- [1] J. D. Bekenstein, *Lett. Nuovo Cimento* **11**, 467 (1974).
- [2] D. Christodoulou, *Phys. Rev. Lett.* **25**, 1596 (1970).
- [3] D. Christodoulou and R. Ruffini, *Phys. Rev. D* **4**, 3552 (1971).
- [4] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
- [5] J. Schwinger, *Phys. Rev.* **82**, 664 (1951); L. Parker and J. Tiomno, *Astrophys. Journ.* **178**, 809 (1972); T. Damour and R. Ruffini, *Phys. Rev. Lett.* **35**, 463 (1975).
- [6] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [7] V. de la Cruz, J. E. Chase and W. Israel, *Phys. Rev. Lett.* **24**, 423 (1970); C.V. Vishveshwara, *Nature* **227**, 936 (1970); M. Davis, R. Ruffini, W. H. Press and R. H. Price, *Phys. Rev. Lett.* **27**, 1466 (1971).
- [8] W. H. Press, *Astrophys. J.* **170**, L105 (1971).
- [9] E. W. Leaver, *Proc. R. Soc. A* **402**, 285 (1985); A. Bachelot and A. Motet-Bachelot, *Ann. Inst. H. Poincaré* **59**, 3 (1993).
- [10] J. D. Bekenstein and V. F. Mukhanov, *Phys. Lett. B* **360**, 7 (1995).
- [11] H-P. Nollert, *Phys. Rev. D* **47**, 5253 (1993).
- [12] N. Andersson, *Class. Quantum Grav.* **10**, L61 (1993).
- [13] V. Mukhanov, *JETP Lett.* **44**, 63 (1986).
- [14] Ya. I. Kogan, *JETP Lett.* **44**, 267 (1986); M. Maggiore, *Nucl. Phys. B* **429**, 205 (1994); C. O. Lousto, *Phys. Rev. D* **51**, 1733 (1995); Y. Peleg, *Phys. Lett. B* **356**, 462 (1995); J. Louko and J. Mäkelä, *Phys. Rev. D* **54**, 4982 (1996); A. Barvinsky and G. Kunstatter, *Phys. Lett. B* **389**, 231 (1996); J. Mäkelä, e-print gr-qc/9602008; H. Kastrup, *Phys. Lett. B* **385**, 75 (1996).