Gravitational Critical Phenomena in the Realm of the Galaxies and Ising Magnets

D. Hochberg and J. Pérez–Mercader

Laboratorio de Astrofísica Espacial y Física Fundamental

Apartado 50727

28080 Madrid, Spain

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Abstract

In the non-relativistic and quasi-static limit, it is possible to map exactly the system of galaxies in the observable universe onto an Ising magnet. Techniques from the theory of critical phenomena as applied to magnets can then be employed to calculate rigorously the galaxy-to-galaxy correlation function, whose critical exponent is predicted to be between 1.530 to 1.862, to be compared to the empirical/observational value of 1.6 to 1.8

The way in which galaxies are distributed at large scales in the observable Universe is codified by the so-called galaxy–to–galaxy correlation function $\xi(r)$, empirically determined to be of the form $\sim r^{-\gamma}$, where r is the distance between galaxies and the exponent $\gamma \sim O(1.6\,\text{to}\,1.8)$ [1,2]. This empirical function plays an important role in cosmology as it provides a means for quantifying the departure from homogeneity and randomness, i.e., is a direct measure of the way the large scale structure in the Universe organizes itself [3]. Any theory purporting to provide a first principles explanation of this large scale structure must account for the form of this function and its attendant exponent. It is the purpose of this

essay to demonstrate how this may be done on the basis of simple Newtonian gravity using well-established results from the theory of critical phenomena [4].

Let $\rho(\mathbf{r})$ denote the spatial density distribution of galaxies, then, the departures from perfect homogeneity are given in terms of $\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \bar{\rho}$ where $\bar{\rho}$ is the average density. The joint probability distribution for finding a galaxy at position $\mathbf{r}_i$ given that there is one at $\mathbf{r}_j$ is $\xi_{Gal}(\mathbf{r}_i - \mathbf{r}_j) =< \delta\rho(\mathbf{r}_i)\delta\rho(\mathbf{r}_j) > /\bar{\rho}^2$, the angular brackets denoting a suitable averaging over large samples of galaxies [3]. Heretofore, this function has only been inferred numerically from observational data as summarized, for example, in galaxy catalogs. As we are not interested here in the internal structure, or morphology, of the individual galaxies, we shall regard the collection of galaxies as a system of discrete, spatially localized masses. In the limit of slow background expansion and weak gravitational field, the energy of the system is dominated by the non-relativistic Newtonian gravitational interaction energy

$$H_{int} = \frac{-1}{2}{\sum_{i,j}}' \frac{m_i\, G\, m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{1}$$

where m_j is the mass of the galaxy at site $\mathbf{r}_j$, and the prime indicates that the sums are over $i \neq j$. The overall expansion, or Hubble flow, of course holds the system of galaxies apart and prevents the otherwise imminent gravitational collapse. Note that in this view of the realm of the galaxies, the density contrast function $\delta\rho(\mathbf{r}_i)/\bar{\rho}$ is equal to 1 if there is a galaxy at $\mathbf{r}_i$ and is equal to -1 if there is a void or local deficit at $\mathbf{r}_i$. Since the observed relative mass dispersion for galaxies is $\delta m/m = 10^{-4}$, it is quite reasonable to assume equal mass galaxies with a common mass which we denote by m_o. It follows that one can establish the following natural mapping between the m_i and a double-valued ($\pm$) "spin"-variable σ_i:

$$m_i = \frac{m_o}{2}(\sigma_i + 1). \tag{2}$$

A galaxy at site $\mathbf{r_i}$ corresponds to spin $\sigma_i = 1$ (spin-up) while a void at site $\mathbf{r}_i$ corresponds to $\sigma_i = -1$ (spin-down). This mapping between spatially localized galaxies and spins is the key to the connection between cosmological large scale structure and spin models for magnets which we now develop. Using it, together with H_{int}, it is possible to write down the

spin-language partition function for the ensemble of gravitationally interacting equal-mass galaxies in the quasi-static limit:

$$Z_{Ising}^{Grav}[\beta] = \sum_{\{\sigma_i\}} exp\left(\frac{1}{4}\sum_{i,j}\sigma_i\, A_{ij}^{-1}\,\sigma_j + \frac{1}{4}\sum_j \sigma_j h_j\right), \tag{3}$$

where $A_{ij}^{-1} = \frac{\beta G m_o^2/2}{|\mathbf{r}_i - \mathbf{r}_j|}$, is the spin-spin coupling and $h_j = \sum_i' \frac{\beta m_0^2 G}{|\mathbf{r}_i - \mathbf{r}_j|}$. This partition function is recognized as one belonging to a spin system in three dimensions coupled to an ambient magnetic field h_j at site j produced by the remainder of the spins. Because of the exact correspondence in (2), the statistical mechanical *and* critical behavior of this equivalent spin model must coincide with that for the gravitationally interacting galaxy system. For the purposes of extracting the correlation function and its critical behavior, it proves convenient to have the above discrete partition function cast in terms of *continuous* random variables. This may be achieved exactly by means of the Hubbard-Stratonovich (or gaussian or Laplace) transformation, [5], which transforms (3) into a path-integral for a continuous field theory:

$$\begin{aligned} Z_{Ising}^{Grav}[\beta] &= C\int[d\phi]\,\exp[-\frac{\beta}{2}\int(\phi(\mathbf{r}) - h(\mathbf{r}))A(\mathbf{r},\mathbf{r}')(\phi(\mathbf{r}') - h(\mathbf{r}'))d\mathbf{r}d\mathbf{r}' \\ &\quad + \int d\mathbf{r}\log\cosh(\beta\phi(\mathbf{r}))] \\ &\equiv \int[d\phi]\, e^{-\beta\mathcal{H}[\phi,h]}, \end{aligned} \tag{4}$$

As is well known [6], the connected two-point function for the spin system (3) and the field theory (4) are identical. Moreover, the random fluctuations in the field ϕ lead to an anomalous dimension η that shifts the canonical dimension of the field for $|\mathbf{r} - \mathbf{r}'| \to \infty$. The *calculated* scaling behavior for the spin-spin (=galaxy–galaxy) correlation function is [7]

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle = \lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\xi_{Gal}(|\mathbf{r} - \mathbf{r}'|) \sim |\mathbf{r} - \mathbf{r}'|^{-(d-2+\eta)}. \tag{5}$$

Here d is the dimension of space ($d = 3$) and η is the critical exponent for the pair correlation function whose value $(0.0198 - 0.064)$ differs from zero due to the *fluctuations* in ϕ. Thus, for a static background, we predict the exponent $\gamma_{static} = (1+\eta)$ to be in the range $1.0198 \leq \gamma_{static} \leq 1.064$

The above calculation has been performed for a quasi-static ensemble of galaxies. But the Universe is expanding and the effects of this expansion will modify the values of the critical exponents. From the theory of dynamical critical phenomena, we know that time enters into the correlation function by *a modification of its argument* [8], viz., the correlation function of Eq. (5) $\xi(r)$ is modified to $\xi(r;t)$, according to the rule

$$\xi(r) \to \xi(r;t) = F(r/l(t)), \tag{6}$$

where $l(t) \sim t^{\zeta}$ is a function of time and r denotes the comoving coordinate, $r_{comoving}$, which is related to the physical coordinate via

$$r_{physical} = a(t)\, r_{comoving}, \tag{7}$$

with $a(t)$ the scale factor for the background spacetime. Depending on whether or not the order parameter is conserved, the non–equilibrium hamiltonian from which Eq. (1) derives by diffusion or relaxation, corresponds to a so–called Model B or A. In both cases one may show (Ref. [9]) from the equation describing the evolution of the order parameter that $\zeta = 1/3$ for Model B and $\zeta = 1/2$ for model A.

For a matter dominated expansion, the case we are interested in, $a(t) \propto t^{2/3}$, and thus for a given physical separation, the relation between time and the comoving coordinate is

$$t \sim (r_{comoving})^{-3/2}. \tag{8}$$

This means that the function $l(t)$ scales as

$$l(t) = t^{\zeta} = (r_{comoving})^{-3\zeta/2}. \tag{9}$$

Putting together these facts in (5) and (6) yields the scaling behavior of $\xi(r;t)$:

$$\begin{aligned} \xi(r;t) &= \left(r_{comoving}/r_{comoving}^{-3\zeta/2}\right)^{-(d-2+\eta)} \\ &= (r_{comoving})^{-(d-2+\eta)(1+3\zeta/2)}. \end{aligned} \tag{10}$$

In other words, the critical exponent resulting from taking into consideration, both the expansion of the Universe and dynamical critical phenomena effects, is

$$\gamma = (d - 2 + \eta) \times (1 + 3\zeta/2). \tag{11}$$

This is our key result. The *predicted* (calculated) value for γ is between 1.530 and 1.596 for $\zeta = 1/3$ (for a conserved order parameter) and between 1.75 to 1.862 for $\zeta = 1/2$ (non-conserved order parameter). These values are to be compared with the values inferred from current galaxy catalogs, which range from 1.5 for the APM survey, to 1.8 for the Lick survey [2,1].

We have thus accomplished what we set out to do, namely, provide a first-principles calculation of the galaxy-galaxy correlation function, of central importance in cosmology, using standard methods of statistical mechanics and the theory of critical phenomena, the language suitable for treating magnets and other states of matter. Naturally, one might (and should) ask *why* this calculation is successful, in other words, what does a system of galaxies have to do with a spin model or a magnet? The key point that justifies our calculation of the critical exponents is the universality hypothesis, according to which rather disparate physical systems will exhibit *identical* critical behavior provided the systems in question possess the same space dimensionality and have order parameters with the same dimensionality (or number of independent components). We see that this is the case here: $d = 3$ for both systems, while the scalar order parameter associated with the density contrast and the magnetization are one-component order parameters for the galaxy system and the 3d Ising model, respectively. In other words, both systems belong to the same universality class. This fact allows one to study the critical behavior of complicated real systems (expanding gas of gravitationally coupled galaxies) in terms of simple model hamiltonians (two component spins on a lattice). Thus, we are *rigorously* led to expect them to have the same values for their critical exponents.

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TABLES

TABLE I. Calculated values for the galaxy-galaxy correlation function critical exponent

Method of Calculation	γ_{Static}	$\gamma^{C}_{Expanding}$	$\gamma^{NC}_{Expanding}$
Series estimates	1.056 ± 0.008	1.584 ± 0.012	1.848 ± 0.014
$O(\epsilon)$	0	1.5	1.75
$O(\epsilon^2)$	1.0198	1.530	1.785
$O(\epsilon^3)$	1.037	1.555	1.815
$O(\epsilon^4)$	1.029	1.543	1.801