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SINGULARITIES

IN

SPACE-TIME

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The conditions under which Einstein's theory of gravity predicts singularities of space-time are examined. It seems that they might all reasonably be expected to be satisfied. The alternatives are then discussed : that the theory breaks down or that singularities actually occur and might be observable.

The prediction of singularities by a physical theory is usually taken as an indication that the theory has broken down. Thus the question whether Einstein's theory of gravitation predicts singularities of space-time is clearly of fundamental importance. On the other hand, even if the theory were to predict singularities, it might be that we should not be in too much of a hurry to discard it until we have observational evidence of whether or not such singularities actually occur.

It should be made clear that, in this essay, the term singularity will be used not for coordinate singularities which are merely the result of a bad coordinate system and can be removed by transformation to another set of coordinates, but for irremovable physical singularities. A precise definition may be given as follows: a space-time will be said to be singularity-free if it is geodesically complete (thatis, every geodesic can be extended indefinitely) and has a C^2 metric tensor field of Lorentz signature. (For the theorems to be described this could probably be weakened to piecewise C^2) Geodesic completeness is necessary as otherwise one could always obtain a non-singular manifold by simply cutting out the bad points and saying that they did not belong to space-time.

It is well known that certain classes of solutions of Einstein's equations exhibit singularities.[1-5] It was felt that these might be special cases and that general solutions would not have singularities.[6] However, recent work by Penrose[7] and by the author[8] has shown that space-time cannot be singularity-free if certain general conditions are satisfied. There are several alternative sets of these conditions. The set given by Penrose is:

1. The Einstein equations hold, i.e.

$$R_{ab} - \frac{1}{2} g_{ab} R + \lambda g_{ab} = T_{ab}$$

where T_{ab} is the energy-momentum tensor of matter.

2. The energy density in the rest frame of any observer, $E = T_{ab} U^a U^b$ is non-negative, where U^a is the unit velocity vector of the observer.

3. There exists a non-compact Cauchy surface. (We define a Cauchy surface to be a complete smooth space-like surface that intersects every time-like and null line once and once only.)

4. There exists a closed trapped surface. (A closed trapped surface is a closed space-like two-surface such that both the congruences of null geodesics which intersect it orthogonally are converging.)

This set of conditions is particularly adapted to prove the occurrence of a singularity in a collapsing star. An alternative set adapted to prove the occurrence of a singularity as a consequence of the expansion or contraction of the whole universe has been given by the author:[8]

1. As for Penrose
2'. $E \geqslant \frac{1}{2}T + \lambda$ where $E = T_{ab}U^a U^b$ as in condition (2) and $T = T^a{}_a$.
3' There exists a surface $\mathcal{H}$ which is a Cauchy surface or which is a complete compact smooth space-like surface.
4' The expansion of V^a the unit normal to $\mathcal{H}$ has a positive lower bound on $\mathcal{H}$ i.e. $V^a{}_{;a} \geqslant C > 0$

The aim of this essay is to examine these various conditions and also to consider the possibility that singularities might actually occurr and if so whether they would be observable.

<u>Conditions 3) and 3')</u>

Solutions of the Einstein equations are known which do not have a Cauchy surface. For example, N.U.T. space,[9] plane-waves[10], Godel's solution,[11] and the anti-De Sitter space[12]. However none of these are very physical.

of course Penrose's theorem only applies to 'open' universes with non-compact Cauchy surfaces. However, if the universe is'closed', condition 3' is a very weak requirement - merely that there exist a complete space-like surface. It may be that our universe does not fulfill conditions 3) or 3') but this is a rather unsatisfactory way out since there will be a general class of solutions with compact complete space-like surfaces, which, if they satisfy the other conditions, must have singularities. Thus it would seem rather arbitrary that our universe happened to be of just such a type as to avoid singularities.

Conditions 4) and 4')

One would expect a closed trapped surface to occur whenever a star passes inside its Schwarzchild radius. As there does not seem to be any mechanism for supporting a cold slowly rotating star of greater than the Chandrasekhar mass, one might expect this to occur frequently . Even if it normally happened that a star exploded as a supernova or lost mass before reaching this stage, one could in theory take a large enough lump of iron and let it collapse.

A closed trapped surface will exist also if the universe is approximately homogeneous and isotropic.[13]

One would expect condition 4' to be satisfied in a universe expanding everywhere. Indeed it might serve as a definition of an expanding universe.

Here again it might be that conditions 4) or 4') are not satisfied by our universe. However, there will clearly be general classes of solutions which do satisfy them.

Conditions 2) and 2')

Condition 2) requires that the energy density is non-negative for every observer. This is true for any matter of which we know and also holds for all the equations of state which have been proposed for super-dense matter. Indeed if matter with negative energy density did exist, there might be difficulties with quantum mechanics since there might not be anything to prevent the creation in a given four-volume of an infinite number of quanta of negative energy and corresponding infinite numbers of positive energy.

Condition 2') is slightly stronger than 2). However, if $\lambda \leqslant 0$ it would be satisfied by a fluid with density μ and isotropic pressure p if $\mu \geqslant 0$, $p \geqslant -\frac{1}{3}\mu$. It would be satisfied by all known matter and all projected equations of state for super dense matter. Even if λ were positive, it would not be very much greater than the mean cosmological density since otherwise the acceleration parameter would be larger than it is observed to be. Thus if the density in the past were several times its present value (and observations indicate that this was so), condition 2') would have been satisfied and there would have been a singularity despite the λ term.

Although normal matter has a positive energy density overall, quantum mechanics would allow negative energy fluctuations over a distance of order of 10^{-14} cms. Thus when the radius of curvature became of the order of 10^{-14} cms., conditions 2) and 2') might no longer hold. However, a radius of curvature of 10^{-14} cms. would be singular enough for most purposes.

Condition 1) The Einstein Equations

It may be that the Einstein equations do not hold for strong fields. So far they have only been experimentally verified for very weak fields. However if we wish to maintain a relativistic theory of gravitation, there are certain difficulties in finding an acceptable alternative set of equations. The left-hand side of the Einstein equations is the only symmetric two index tensor expression involving no derivatives of the metric higher than the second, linear in second derivatives and having zero divergence. It may be derived by variation of the Lagrangian density $R\sqrt{-g}$. This might suggest that one should try some other Lagrangian density such as $R_{ab}R^{ab}\sqrt{-g}$ or $R_{abcd}R^{abcd}\sqrt{-g}$. However these all lead to expressions involving fourth order derivatives of the metric. This would mean that in order to determine the time development of a region one would have to know the initial values of g_{ab} and its first, second and third derivatives. This is contrary to all our experience that we only need to know the state of a system and its first derivative to determine its time development. Of course it could be that the higher derivatives only occurred in second order terms (i.e. multiplied by the curvature) so that their effect would be undetectable in the weak fields of which we have experience. Nevertheless it is a rather unattractive possibility and might lead to absurd results like the 'runaway' solutions which arise in electro-dynamics when the radiation reaction term, which involves third derivatives, is taken literally.

Another possibility would be that one should abandon trying to derive the field equations from a Lagrangian and should simply add terms in the Riemann tensor to the left-hand side of Einstein's equations. One might try terms like $R_a{}^{cde}R_{bcde}$. However the left-hand side would then no longer have zero divergence. Indeed it is difficult to think of any such expression involving only second derivatives of the metric which would have zero divergence. This would mean that the energy-momentum tensor could no longer have zero divergence and there would not be local conservation of energy. There might be an additional complication as follows: the ten Einstein equations are not all independent because there are the four divergence relations connecting them. They are, however, sufficient since the ten components of g_{ab} cannot be completely determined by the field equations as there must always be four degrees of freedom to make coordinate transformations. If, however, one had a left-hand side which did not satisfy any identities, there would be ten independent equations and the system would be overdetermined unless the energy momentum tensor satisfied only six independent equations of state. These would have to be compatible with the left-hand side and this might be difficult to arrange. For example, empty space or pure electromagnetic field would not be compatible.

Of course the Einstein equations are only a classical description of the gravitational field. Presumably it would be necessary to consider quantum effects in regions of strong curvature. However the length scale for this seems to be 10^{-33} cms. which is even smaller than that in the discussion on condition 2).

Singularities

Maybe, after all, singularities do occur. If so, what will they be like? Unfortunately the methods of Penrose[7] and the author[8] do not give any information on this. They merely show that space-time cannot be singularity free. Some idea of the possible types of singularities may be gained by studying the known solutions with singularities. For instance the Friedmann solutions have singularities that intercept every time-like line and the Schwarzchild solution has singularities which intercept every time-like line going inside the Schwarzchild radius. However the Reissner-Nordstrom and Kerr solutions have singularities that most time-like lines manage to avoid.-by going through a 'wormhole' and emerging in another universe. One might expect that a collapsing star that was rotating or had a charge excess would give rise to singularities like those in the Kerr or Reissner-Nordstrom solutions respectively. Thus a collapsing star might create a 'wormhole' into another universe. Unfortunately it is a consequence of Penrose's theorem that we who stay behind can never see the singularity or the wormhole. The only way to find out whether there were a wormhole would be to dive in after the collapsing star and then one would be unable to return to tell anyone else. If one emerged into the other universe, one might hope by diving into another collapsing star in that universe to find a wormhole leading back to our universe. However this possibility could probably be ruled out on the grounds that it would lead to closed time-like lines and violation of causality.

On the other hand the singularities in the past predicted by the author's theorems[8] should in principle be observable. They need not have the all-embracing character of the Friedmann solutions. They might be more like the Kerr singularities. However it might be difficult to observe them in practice as they might be surrounded by clouds of ionised gas or other matter which might be entering the universe at the singularity. This indeed has been a suggested explanation of quasars.

Although it might be difficult to observe a singularity, it might be possible to make a negative observation that no singularity occurred. If an extension of present observations were to indicate that, prior to the present expansion, there was a contracting phase and that the switch-over occurred without any singularities, then this would show that either the Einstein thory does not hold or that one of the other conditions is not satisfied.

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