A NEW HAMILTONIAN STRUCTURE FOR THE DYNAMICS OF GENERAL RELATIVITY

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Abstract

A new compact form of the dynamical equations of relativity is proposed. The new form clarifies the covariance of the equations under coordinate transformations of the spacetime. On a deeper level, we obtain new insight into the infinite dimensional symplectic geometry behind the equations, decompositions of gravitational perturbations and the space of gravitational degrees of freedom. Prospects for these results in studying fields coupled to gravity and the quantization of gravity is outlined.

It has been over fifteen years since Arnowitt, Deser and Misner laid down the basic formulation of Einstein's equations as dynamical equations for an evolving spatial universe ([3], [16]). This procedure is basic for establishing the existence and uniqueness theorems in general relativity ([6], [14]), for a study of stability of spacetimes ([8], [12], [19]) and the positivity of their energy content ([4], [9]), and for approaching many other important questions.

On the other hand, there have been significant developments in Hamiltonian mechanics and symplectic geometry (i.e. Poisson bracket structures) during the same period. Both classical mechanics and field theories have been successfully put into this general context ([1], [5].) From this geometrization and unification spring new insights and methods. Specifically, there is now a satisfactory general procedure for eliminating the symmetries of a given Hamiltonian system. Previously, this was well understood in the classical literature only for commutative symmetry groups (i.e., for the relatively rare occurrence of first integrals in involution) and in special cases, such as rotational symmetry (see [15] and [20]). This is important in relativity since its gauge group is non-commutative and is infinite dimensional.

It is tempting to apply this new elimination procedure to the space of solutions to Einstein's equations and thereby remove the coordinate symmetries of a spacetime. The resulting space of four-geometries is called the space of gravitational degrees of freedom. Some important conformal representations of this space have been constructed by York ([19], [21], [22]). However, we desire a construction which is natural with respect to the dynamics, and wish to prove that the space is a smooth infinite dimensional manifold and carries a Poisson bracket structure.

In the meantime, Moncrief [18] has published an important new decomposition of gravitational perturbations, one piece of which represents the direction of the space of gravitational degrees of freedom. This decomposition unifies and extends several previous decompositions in geometry and relativity due to Barbance, Deser and Berger-Ebin. However, only Moncrief's formulation reveals explicitly the symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad I = \text{identity operator.}$$

The above developments suggest that between these otherwise diverse approaches some beautiful connections can be made. However, there is an immediate and basic obstacle. The dynamical equations of Arnowitt-Deser-Misner are not written in a form which makes explicit use of the symplectic structure and consequently it is not clear how to use the symplectic geometry alluded to above. Part of the problem is that the equations contain terms involving the lapse function N and the shift vector field X. These terms are necessary because of the arbitrary way the spacetime can be sliced into space and time or, in other words, because of the coordinate invariance of the spacetime.

The remainder of the essay will explain how the authors have solved this problem using a new form for the evolution equations and how the solution might be useful for future research.

We begin with some notation that will enable us to state the equations. Let M be a 3-manifold and let Q be an open subset of a linear space of C^∞ tensor fields of some specified type on M and let Q^* be the space of dual tensors. For instance, if Q consists of the symmetric covariant two tensors $\phi = \phi_{ij}$, Q^* is the set of contravariant symmetric two tensor densities $\pi = \pi^{ij}$. If $L: Q_1 \to Q_2$ is a linear differential operator, $L^*: Q_2^* \to Q_1^*$ denotes its adjoint obtained in the usual manner by integration by parts. If $T: Q_1 \to Q_2$ is a non-linear differential operator, $DT(\phi)$ denotes its linearization (= Fréchet derivative, or functional derivative) at $\phi \in Q_1$, so $DT(\phi)$ is a linear differential operator from Q_1 to Q_2. We let J denote the symplectic matrix on $Q \times Q^*$ as defined above and let $P = Q \times Q^*$ denote the <u>phase space</u>.

Lie differentiation $\mathcal{L}_X\phi$ of fields ϕ by vector fields X is a first order differential operator in X. Its negative adjoint is called the <u>flux density</u> $\mathcal{J}$; it may be regarded as a map of P to Λ^1_d, the one form densities (dual to vector fields), and is explicitly defined by

$$\int X\cdot\mathcal{J}(\phi,\pi) = -\int \pi\cdot\mathcal{L}_X\phi \quad .$$

Let $\mathcal{H}: P \to C^\infty_d$ (scalar densities) be a given Hamiltonian density and define

$$\Phi(\phi,\pi) = (\mathcal{H}(\phi,\pi),\mathcal{J}(\phi,\pi)) \ .$$

For general relativity, Q is the space of Riemannian metrics g_{ij} on M, Q^* the symmetric two tensor densities π^{ij} and (see [3], [4], [10])

$$\mathcal{H}(g,\pi) = (\pi\cdot\pi - \frac{1}{2}(\mathrm{tr}\pi)^2 - R(g))\sqrt{\det g_{ij}}$$

where $\cdot$ denotes contraction to scalars, tr is trace and $R(g)$ is the scalar curvature. One calculates that $\mathcal{J}(g,\pi) = 2\pi_i{}^j{}_{|j}$, twice the covariant divergence of π.

Let us next recall the meaning of the lapse and shift functions of Wheeler (see [16]). Let V be a spacetime with a Lorentz metric ${}^{(4)}g$. Let i_λ be a <u>slicing</u> of V by M;

i.e., for each number λ , i_λ is an embedding of M to a space-like hypersurface of V (and these embedded manifolds fill out an open set in V). The λ-derivative of i_λ is a vector field on V defined along the imbedded hypersurfaces. Its normal and tangential components, regarded as scalar and vector functions on M , are called the <u>lapse</u> N and <u>shift</u> X . They depend of course on the slicing of the spacetime and in fact characterize the slicing.

For vacuum spacetimes, Einstein's equations state that the Ricci tensor of ${}^{(4)}g$ vanishes. Arnowitt-Deser-Misner showed that these equations are equivalent to certain complicated looking evolution equations and constraint equations (see [16, p. 525]) for the 3-metric g_{ij} induced on M by a slicing and its corresponding conjugate momentum π^{ij} (defined to be $((k^\ell{}_\ell)g^{ij}-k^{ij})\sqrt{\det g_{\ell m}}$ where k_{ij} is the second fundamental form or extrinsic curvature of the embedded hypersurface regarded as a two tensor on M).

Our first main point is that these equations can in fact be written in the following compact way (λ , the slicing parameter is often denoted t , but it need not be a time-like direction so we use λ):

$$\text{(E)}\quad \frac{\partial}{\partial\lambda}\begin{pmatrix} g \\ \pi \end{pmatrix} = J\cdot D\Phi(g,\pi)^* \begin{pmatrix} N \\ X \end{pmatrix} \qquad \text{[evolution equations]}$$

$$\text{(C)}\quad \Phi(g,\pi) = 0 \qquad \text{[constraint equations]} .$$

Computing the adjoint $D\Phi(g,\pi)^*$ (see [12]) shows that these equations are equivalent to the Arnowitt-Deser-Misner equations.

This new way of writing the equations is of intrinsic interest in itself. However, we claim something more profound: we assert that equations of the same form also apply if there are general tensor fields present (for example, electromagnetic or matter fields) in addition to the gravitational fields. These new equations are

$$(E_T) \quad \frac{\partial}{\partial\lambda}\begin{pmatrix} g, & \phi \\ \pi, & \pi_\phi \end{pmatrix} = J\cdot D\Phi_T((g,\phi),(\pi,\pi_\phi))^* \begin{pmatrix} N \\ X \end{pmatrix}$$

$$(C_T) \quad \Phi_T((g,\phi),(\pi,\pi_\phi)) = 0$$

where ϕ represents all fields other than the metric g, π_ϕ is the conjugate momentum of ϕ and $\Phi_T = (\mathcal{H}_{\text{relativity}} + \mathcal{H}_{\text{other fields}}, \mathcal{J}_{\text{relativity}} + \mathcal{J}_{\text{other fields}})$ is the total Hamiltonian and flux density for the coupled system. These equations give a unified Hamiltonian formulation of general field theories coupled to gravity!

We shall now make a series of remarks intended to show the geometric and analytic utility of this new formulation of Hamiltonian equations for field theories.

First of all, the form shows explicitly how the dynamical equations are generated by the constraints, and how the equations depend explicitly on the slicing. Moreover, it shows that the equations are of Hamiltonian type (see [5]) using a symplectic structure J independent of the slicing.

Next, the form (E) allows one to see more easily relationships between properties of the spacetime and corresponding conditions on the Cauchy data. For example, it simplifies the calculations in the proof of Moncrief's criteria, which relates linearization stability of solutions of the constraint equations, and hence of the spacetime, to the absence of Killing fields on the spacetime (see [8], [12], [17]). Very recently this idea has been used by J. Arms to successfully analyze the linearization stability of the coupled Einstein-Maxwell system.

Thirdly, it gives a unified picture of decomposition theorems used in relativity. Moncrief's basic decomposition theorem [18] states that the phase space can be decomposed as follows:

$$P = \text{range } J\cdot D\Phi(g,\pi)^{*} \oplus (\text{kernel } D\Phi(g,\pi) \cap \text{Kernel } D\Phi(g,\pi)\cdot J) \oplus \text{range } D\Phi(g,\pi)^{*} = ① \oplus ② \oplus ③ \qquad \text{(D)}$$

(see Figure 1 below). This generalizes Deser's classical decomposition of tensors into transverse-traceless and other pieces. In terms of the new equations (E), the decomposition (D) becomes a special case of a general fact in symplectic geometry ([2]). The present formulation is not merely a restatement; it also shows us with no extra effort how to explicitly decompose perturbations of general field theories coupled to gravity!

Finally, the form (E) enables us to give a representation of the space of gravitational degrees of freedom which is directly related to the dynamical equations. We let $\mathscr{C}$ denote the space of solutions of the constraint equations (C) . It is known under what conditions $\mathscr{C}$ is a manifold near (g_0, π_0) (see [12]). A family i_λ of embeddings of M into V , i.e., a slicing, acts on the space $\mathscr{C}$ as follows: if $(g_0, \pi_0) \in \mathscr{C}$ and i_λ is an embedding of M in V , then (g_0, π_0) is transformed to (g_λ, π_λ) ; the metric and conjugate momentum induced on the hypersurface $i_\lambda(M)$ in the four-geometry ${}^{(4)}g$ generated by (g_0, π_0) on $i_0(M)$. This transformation is exactly that induced by the dynamics (E) and is therefore an (infinite dimensional) canonical transformation. If we identify all such (g_0, π_0) and (g_λ, π_λ) we obtain a quotient space $\mathcal{S}$. The general theory of reduction of phase spaces with symmetry [15] shows that $\mathcal{S}$ is a smooth symplectic manifold. Moreover, since coordinate transformations yield the different possible slicings, $\mathcal{S}$ is identifiable with space of solutions of Einstein's equations with solutions related by a coordinate transformation identified; i.e., with the space of gravitational degrees of freedom.

The tangent space to $\mathscr{E}$ is exactly the second summand in the decomposition (D) showing the natural relationship of the two ideas. The three summands in (D) and the manifolds $\mathscr{C}$ and $\mathscr{E}$ are shown in Figure 1.

Similar methods of symplectic geometry can be applied to give results for general field theories coupled to gravity. Our new formulation of these coupled systems allows for the organization of deep theorems concerning the structure of the spaces of degrees of freedom in a systematic and unified manner.

Future prospects for the methods described here are bright. There is every reason to believe that a more profound understanding of fields coupled to the purely gravitational field will result. In another direction, there is hope that it will help clarify the quantum gravity problem as well. Admittedly, the solution of the coordinate gauge problem is only a beginning but its rigorous resolution is still a significant one, for we now have a well defined symplectic space in which to quantize.

It is gratifying that methods of infinite dimensional analysis have been so successful in recent years (see [4], [7], [9], [11], [14]). It is now time to seriously use the additional machinery provided by the natural symplectic structure of the spaces involved.

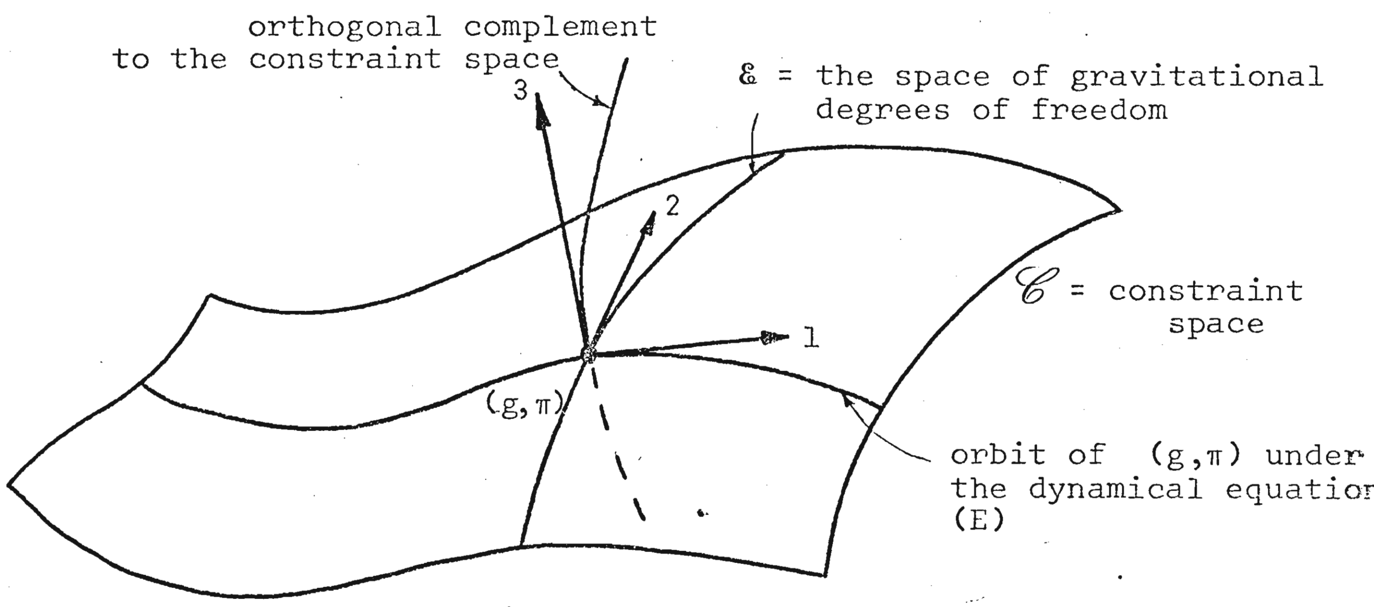
orthogonal complement
to the constraint space
3
2
1
ℰ = the space of gravitational
degrees of freedom
𝒞 = constraint
space
(g, π)
orbit of (g,π) under
the dynamical equation
(E)

Figure 1.

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Dear Mr Rideout

The following is the biographical information to go with our entry in the essay contest (in a separate envelope):

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Sincerely

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