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Global Analysis and General Relativity.

Arthur E. Fischer and Jerrold E. Marsden

Abstract

An outline of recent applications of modern infinite dimensional manifold techniques to general relativity is presented. The uses, scope, and future of such methods are delineated. It is argued that the mixing of the two active fields of general relativity and global analysis provides stimulation for both fields as well as producing good theorems. The authors' work on linearization stability of the Einstein equations is sketched out to substantiate the arguments.

The past few years have seen new branches of mathematics applied to problems in general relativity. One of the most important of such applications has been to the study of the topology of spacetimes in the works of Geroch, Hawking and Penrose. Using techniques of differential topology and differential geometry, they prove, for example, various incompleteness theorems from which one may infer the existence of black holes--under reasonable mathematical hypotheses on the spacetime involved. See W. Kundt [11] for a recent survey and a bibliography for this subject .

The techniques used in the above are taken from the study of the topology and geometry of finite dimensional manifolds. However the theory of infinite dimensional manifolds has been considerably developed over the past fifteen years and the time is ripe for their application to general relativity. The purpose of this essay is to outline some applications which have been made and to point out some directions for future work in this field.

There have already been some significant applications of the theory of infinite dimensional manifolds to other fields. Perhaps the first of these was given by Eells, Palais and Smale to the calculus of variations (cf. Smale [16]). Their ideas and methods have been a great stimulation to other workers in non-linear analysis. Another application has been given to fluid mechanics by Ebin and Marsden [6]. In these applications, analysis on the infinite dimensional spaces involved is not superficial, consisting merely of a rehash of old ideas in fancy language. Rather, the methods reflect a fundamental change in policy, with the new analysis being used in an essential way.

That infinite dimensional manifold theory is relevant for general relativity was first pointed out by J. A. Wheeler (cf.[17]). He stressed the usefulness of considering superspace $\mathcal{S}$. $\mathcal{S}$ consists of riemannian metrics on a given three manifold M, with metrics which can be obtained one from the other by a coordinate transformation, identified. This space $\mathcal{S}$ is important for we can view the universe as an evolving (or time dependent) geometry and hence as a curve in $\mathcal{S}$. The geometry and topology of $\mathcal{S}$ has been investigated by several people. See for example Fischer [6].

Recall that the Einstein equations of general relativity state that, outside of regions of matter, the metric tensor $g_{\alpha\beta}$ must be Ricci flat; i.e. $R_{\alpha\beta} = 0$. This is a complicated coupled system of non-linear partial differential equations. One can regard the Einstein equations as a Hamiltonian system of differential equations on $\mathcal{S}$ in an appropriate sense. This idea goes back to Arnowitt, Deser and Misner [1] but was put into the setting of $\mathcal{S}$, explicitly using infinite dimensional manifolds by Fischer-Marsden [8].

The above applications to general relativity can be regarded as "soft" in the sense that infinite dimensional manifolds are involved mostly as a language convenience and as a guide to the theory's structure. While this is important, it is perhaps not critical to the development of the theory.

The first substantial "hard" theorem using infinite dimensional analysis (at least in an informal way) is due to Brill and Deser [3]. They establish the important result that any non-trivial perturbation of Minkowski space leads to a spacetime with strictly positive mass (or internal gravitational energy). The technique they use is an adaptation of methods from the calculus of variations. The idea behind the proof is rather simple; they show that on the space of solutions to Einstein's equations, the mass function has a non-degenerate critical point at flat, or Minkowski, space. Their investigations have inspired a number of recent results ; cf.[10], [13] and [15].

An important feature of the work of Brill and Deser is that the infinite dimensional techniques employed are natural, useful and indispensible.

Another fundamental problem in general relativity which has been solved using techniques from global analysis is that of _linearization stability_. This problem may be explained as follows. Suppose we have a given spacetime, for example the Schwarzchild metric, and then wish to consider a slightly perturbed situation; for instance the introduction of a slight irregularity or a small planet. To consider such situations directly is not easy because of the non-linear nature of Einstein's equations. Instead, it is common to linearize the equations, solve these linearized equations, and assert that the solution is an approximation to the "true" solution of the non-linear equations.

It is perhaps surprising that the assumption made--that the solution of the non-linear equations approximates the solution of the full equations--is not always valid. Such a possibility was indicated by Brill [2], and has been established rigorously by the authors in the case the universe is "toroidal"; i.e. $T^3 \times \mathbb{R}$ where T^3 denotes the flat 3-torus. If the above assumption on the given spacetime is valid, that spacetime is called linearization stable.

Our theorem below shows that Brill's example is exceptional and that most spacetimes can be expected to be linearization stable. Although it would be unpleasant if this were not so, the example and the delicacy of the result show that caution is to be exercised when such sweeping assumptions are made.

Theorem. Suppose that the ("background") spacetime with metric tensor $g_{\alpha\beta}$ satisfies the following conditions: there is a space like hypersurface M with induced metric g and second fundamental form k such that

(i) there are no infinitesimal isometries X of both g and k (if M is not compact, X is required to vanish at infinity)

(ii) *if* $k = 0$ *and* M *is compact then* g *is not flat*

(iii) *if* $k \neq 0$, $tr(k)$ = *trace of* k *is constant on* M *if* M *is compact, and* $tr(k) = 0$ *if* M *is non-compact*.

(iv) *if* M *is non-compact*, g *is complete and in a suitable sense asymptotically Euclidean*.

Then near M, *the spacetime metric* $g_{\alpha\beta}$ *is linearization stable*.

Brill's example fits in because condition (ii) fails for $M = T^3$, the flat 3-torus.

The following corollary was obtained by Choquet-Bruhat and Deser [5] independently of the authors.

Corollary. *Minkowski space is linearization stable*.

Although the proof is complicated in details, we can endeavor to give the main ideas here. It is a simple and elegant application of the theory of infinite dimensional manifolds.

In order to solve the Einstein equations, one can regard them as evolution equations with g, k (as given in the statement of the theorem) as initial, or Cauchy, data. However there are some non-linear constraints involved called the divergence constraint, written

$\delta\pi = 0$ and the Hamiltonian constraint $\mathcal{H} = 0$ which g, k must satisfy. This defines a certain non-linear subset $\mathcal{C}$ of $T\mathcal{M}$, the space of all g's and k's on M. The principal method is the following: near those g, k for which the conditions of the theorem are satisfied, the set $\mathcal{C}$ is a smooth infinite dimensional submanifold of the space $T\mathcal{M}$. The other points are singular.

The smoothness of the set $\mathcal{C}$ entails that tangent vectors to $\mathcal{C}$ are closely approximated by points in $\mathcal{C}$ itself (which would not be the case if $\mathcal{C}$ has corners or other singularities). This remark together with existence theorems for the Einstein equations (cf. [4], [9]) yields the desired result.

Fortunately, establishing the smoothness of $\mathcal{C}$ can be done by techniques which have been previously developed in infinite dimensional manifold theory (these are found in, for example, Lang [12]).

We suggest that there are several other problems which can be attacked by the methods of infinite dimensional analysis. Specifically we suggest the following

(a) rigorously establish the claims made in Brill-Deser [3] concerning the <u>global</u> positivity of the mass function

(b) study conditions on initial data which guarantee conditions under which the resultant spacetime will be free of singularities and hence free of black holes (cf. [13])

(c) study stability properties of solutions to Einstein's equations, in the same sense as the solar system is stable in classical mechanics.

It seems, in view of our experience in these matters, that such goals for the immediate application of global analysis techniques are not unreasonable ones.

The techniques of global analysis are appropriate for general relativity because of the non-linear nature of the problems involved. Since the field equations are non-linear, the spaces of solutions will also be non-linear and so infinite dimensional manifold techniques are appropriate for their study. There is a promising future for the development of this bridge between non-linear analysis and general relativity.

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Biographical Information

Arthur E. Fischer

Born 1945 New York City

BA 1965 Columbia University

PhD 1969 Princeton University

Lecturer in Mathematics 1969-1972 University of California, Berkeley,

Assistant Professor of Math 1972-present, University of California, Santa Cruz.

Jerrold E. Marsden

Born 1942 Ocean Falls, B. C. Canada

BSC 1965 University of Toronto

PhD 1968 Princeton University

Lecturer in Mathematics 1968-9 University of Cal, Berkeley.

Assistant Professor of Math 1969-70 " " " "

Assistant Professor of Math 1970-71 Univ. of Toronto

Associate Professor of Math 1972-present, University of Cal, Berk.