STRONG CURVATURE NAKED SINGULARITIES IN NON-SELF-SIMILAR GRAVITATIONAL COLLAPSE

P.S. Joshi and I.H. Dwivedi*
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
India

* Permanent Address:
Institute of Basic Sciences
Agra University
Khandari, Agra, India

Address for Correspondence:
Dr. Pankaj S. Joshi
Theoretical Astrophysics Group
T.I.F.R. Homi Bhabha Road
Colaba, Bombay 400 005
India

Abstract

It is shown that strong curvature naked singularities form in a non-self-similar gravitational collapse of radiation. The imploding radiation space-times with a general form of mass function are analyzed and we show that a strong curvature property holds along all families of non-spacelike geodesics terminating at the singularity in past. In view of the strength of singularity and the non-self-similar nature of space-time, we believe this is a very serious counter-example which must be taken into account for any possible formulation of the cosmic censorship hypothesis.

The conjecture, that nature provides a built-in safety mechanism which covers a space-time singularity formed during gravitational collapse from outside observers, is called the cosmic censorship hypothesis [1]. This remains the most important unresolved problem in classical general relativity, and lies at the foundation of the currently well-accepted and applied theory of black holes. As yet, any attempts for a rigorous mathematical formulation and a proof for the same have not been successful. Hence, the examples which show the occurrence of naked singularities under various space-time situations remain important and must be analyzed carefully and in detail, as these would provide indication for a possible formulation and proof for the censorship hypothesis. Some of these examples describe shell-crossing singularities (see e.g. [2]), but there are others describing shell focusing singularities which are more difficult to ignore. Important examples of shell focusing naked singularities analyzed so far are the dust collapse in marginally bound self-similar Tolman-Bondi models [3], spherical self-similar collapse of an adiabatic perfect fluid with a soft enough equation of state [4], and the linear mass Vaidya solutions for radiation collapse [5]. In fact, a detailed analysis of the radiation collapse scenario has shown that the resulting naked curvature singularity is extremely strong in the sense that curvatures diverge along *all* the families of non-spacelike geodesics terminating at the singularity in past [6].

A feature that emerges from the present scenario is that much of the presently available discussion on naked strong curvature singularities and examples are confined to self-similar spacetimes only. Even though it was implicit in the earlier work [3] that a Vaidya space-time with an initially linear mass function can provide a non-self-similar space-time, somehow the suggestion coming from the analysis so far is that the naked similarity is related to some geometric property of self-similar spacetimes rather than gravitational dynamics of matter therein [3,4]. Now, self-similarity is a kinematic property of a space-time whereas natural formulation of gravitational collapse would be in terms of an initial value problem. Again, self-similar spacetimes are not asymptotically flat and hence do not provide a natural background to model gravitational collapse of compact objects.

It is thus a matter of importance to learn if any serious examples of naked singularities arise in non-self-similar space-time. The answer here would help towards a better formulation of the cosmic censorship conjecture, which is the first task at the moment requiring special attention.

The purpose of this paper is to show that strong curvature naked singularities occur in a non-self-similar collapse of radiation, and to analyze the structure of the same. Our analysis on the structure here will mainly focus on examining the strength of the naked singularity, as this provides a very important indication of the seriousness of a naked singularity [7]. The censorship conjecture, as originally proposed by Penrose [1] emphasizes on the stability criteria for spacetimes. However, such criteria are extremely difficult to formulate and test in general relativity, and hence Newman [8] proposed the alternative formulation of the conjecture that naked singularities must be gravitationally weak. The advantage with this formulation is that various criteria to test the gravitational strength of singularity are available and by applying the same one can analyze the structure of naked singularity to deduce seriousness of a given example and thereby narrow the choice of tenable conjectures.

The imploding radiation is modeled by the Vaidya space-time, given in (u, r, θ, ϕ) coordinates as,

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 + 2dudr + r^2 d\Omega^2 \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The radiation collapses at the origin of the coordinates, $u = 0, r = 0$; where u is advanced time and $M(u)$ is an arbitrary but non-negative increasing function of u. The situation, when M is a linear function of u (i.e. $M(u) = \lambda u$, $\lambda =$ const.) has been studied extensively [5] and it is seen that the radiation shells collapse to form a central singularity which will be naked and persistent if the collapse is sufficiently slow. The energy tensor for the radial flux of radiation is,

$$T_{ab} = \rho k_a k_b = \frac{1}{4\pi r^2}\frac{dM}{du} k_a k_b \tag{2}$$

with $k_a = -\delta_a^u$ and $k_a k^a = 0$. Note that $dM/du \geq 0$ implies that the weak energy condition is satisfied. The Minkowski space-time for $u < 0$, $M(u) = 0$ here is joined to a Schwarzschild space-time for $u > T$ with mass $M(T)$ by way of the Vaidya metric (1).

We note that when $M(u)$ has a linear form, i.e. $M(u) = \lambda u$, $\lambda =$ const, the space-time is self-similar admitting a homothetic killing vector and the earlier conclusions on the formation of a powerful strong curvature naked singularity are recovered. On the other hand, if $M(u)$ has any other non-linear form, the basic requirement of self-similarity (namely, $g_{tt}(ct, cr) = g_{tt}(t, r)$ and $g_{rr}(ct, ct) = g_{rr}(t, t)$) is not satisfied. It is this class of non-self-similar spacetimes with general mass functions which is of interest to us here.

Writing $K^a = dx^a/dt$ as tangent to non-spacelike geodesics, these quantities for a general form of mass function $M(u)$ are given [6] as below:

$$K^u = \frac{du}{dk} = \frac{P(u,r)}{r} \tag{3a}$$

$$K^r = \frac{dr}{dk} = \frac{1 - \frac{2M(u)}{r}}{2r}P - \frac{l^2}{2rP} + \frac{Br}{2P} \tag{3b}$$

$$k^\theta = \frac{l\cos\beta}{r^2\sin^2\theta} \tag{3c}$$

$$k^{\phi} = \frac{1 \sin\beta\cos\phi}{r^2} \tag{3d}$$

where $u = rX, l$ is the impact parameter, β is the isotropy parameter given by $\sin\phi\tan\beta = \cot\theta$ and k is the affine parameter along the geodesics. For radial curves we have $l = 0$. The constant B characterizes different classes of geodesics, i.e. $B = 0$ for null curves, $B < 0$ for timelike curves and $B > 0$ for spacelike curves. The function P satisfies,

$$\frac{dP}{dk} = \frac{P^2}{2r^2}\left(1 - \frac{4M(u)}{r}\right) + \frac{l^2}{2r^2} + \frac{B}{2} \tag{4}$$

The radial null geodesics in such a space-time are obtained from equations (3) and given by,

$$\frac{du}{dr} = \frac{2r}{r - 2M(u)} \tag{5}$$

In case $M(u)_{|u=0} \neq 0$, the singularity is surrounded by an event horizon and is not naked, which is a situation corresponding to an initial mass already present at $u = 0, r = 0$, i.e. shell collapse in a Schwarzschild background. On the other hand the situation here is that of radiation injected into an initially flat and empty region and focused into a central singularity of growing mass by a distant spherical source. The source is turned off at a finite time T when the field settles to a Schwarzschild case. It follows that the differential equation (5) has a singular point at $u = 0, r = 0$. The nature of this singular point can be analyzed by standard techniques[9] and writing

$$2\left(\frac{dM(u)}{du}\right)_{u=0} = \lambda \tag{6}$$

the roots of the characteristic equation $\eta^2 - \eta + 2\lambda = 0$ for the equation (5) are given by,

$$\eta = \frac{1 \pm \sqrt{1 - 8\lambda}}{2} \tag{7}$$

It follows therefore that for $0 < \lambda \leq 1/8$ the singular point becomes a node for the family of radial null geodesics and these curves meet the singularity with a definite tangent. When $\lambda = 0$, the structure of the singularity is somewhat complicated. It is not a pure node but could be a col-node where some characteristics still pass through the singularity which will be naked. This we will discuss here later.

Thus, the singularity in question for the space-times given by (1) is a naked singularity for general mass functions defined by the above conditions. This establishes the occurrence of naked singularities for a class of non-self-similar spacetimes describing radiation collapse. It is of course possible that the singularity can be locally, or globally naked. This can be distinguished by an analysis of the critical direction associated with the node, i.e. examining the associated Cauchy horizon. For example, if $r(T) < 2M(T)$ along the Cauchy horizon, the node is only locally naked. However, we will not elaborate on this because our main purpose here is to show the existence of a naked singularity for non-self-similar gravitational collapse, and then to analyze the strength of the same. It is not important whether the singularity is locally, or globally naked; because either way it violates cosmic censorship though in varying degrees of seriousness. When the singularity is globally naked, its emissions are visible to far away observers and predictability is violated in the space-time. On the other hand, a locally naked singularity can emit non-spacelike trajectories, however such causal influence never reaches an asymptotic observer. Our intention is the analysis of strength in either case.

In the linear-mass Vaidya solution, in addition to the radial null geodesics, other families of non-radial non-spacelike geodesics also terminate at the singular point $r = 0$, $u = 0$ with a definite tangent and it was shown [6] that along all such families of singular geodesics a strong curvature condition [7] is satisfied, namely,

$$\lim_{k\to 0} k^2 R_{ab} K^a K^b > 0 \tag{8}$$

Thus, the resulting naked singularity turns out to be strong in an extremely powerful sense. The details and specification of the non-spacelike families turn out to be quite complicated in the linear-mass Vaidya case itself and we do not intend to discuss here the same for a general mass function in a non-self-similar space-time. However, we present here the proof on the strength of the singularity in this case which will generally apply to any family of non-spacelike geodesics which meet the singular point $u = 0, r = 0$ with a definite tangent, and not just the null radial family. We show that along all such families of singular non-spacelike geodesics, the strong curvature condition (8) is satisfied.

Since for any family of non-spacelike curves meeting the singularity the tangent is definite, it follows that du/dr is well defined at $u = 0$, $r = 0$. Define the quantity X_0 as,

$$X_0 \equiv \lim_{u\to 0, r\to 0} \left(\frac{u}{r}\right) = \lim_{u\to 0, r\to 0} \frac{du}{dr} \tag{9}$$

Using equations (3a) and (3b), this implies

$$X_0 = \frac{2P_0^2}{P_0^2(1 - \lambda X_0) - l^2} \tag{10}$$

where $P_0 = \lim_{u\to 0, r\to 0} P$. In order to evaluate the strength, consider the scalar $\psi = R_{ab}K^a K^b$, where K^a is tangent to non-spacelike geodesics. For the Vaidya space-time (1) we get using (2) and (3a) (k is an affine parameter along the geodesic with $k = 0$ at the singularity),

$$k^2\psi = \frac{2dM(u)}{du} P^2 \left(\frac{k}{r^2}\right)^2 \tag{11}$$

Using equations (3a,b), (4), (6), (10), and the l' Hospital's rule, the limit in (11) can be evaluated along singular geodesics as $k \to 0$ and we get,

$$\lim_{k \to 0} k^2 \psi = \frac{\lambda X_0^2}{4} \qquad \text{for } P_0 \neq \infty \tag{12a}$$

$$\lim_{k \to 0} k^2 \psi = 4\lambda \qquad \text{for } P_0 = \infty \tag{12b}$$

It therefore follows that for any positive value of λ the above limit is always positive and the strong curvature condition (8) is satisfied, except for the case when $X_0 = 0$. However, this last situation corresponds to $P_0 = 0$ at the singularity. It is not difficult to see by integrating the geodesic equation (3) near the singularity that if a non- spacelike geodesic is meeting the singularity in past, this value is not realized along it.

We have thus shown that a serious example of naked singularity arises for a general class of non-self-similar spacetimes, namely the Vaidya solutions with a non-linear mass term, describing the gravitational collapse of radiation at the centre.

We now discuss the case $\lambda = 0$ in some detail, i.e. the derivative $dM(u)/du$ vanishes at the origin. Then, as shown above, the curvature condition (8) is not satisfied. This characterization of the strength implies that in the limit of approach to the singularity, all volume forms along a non-spacelike geodesic are crushed to zero size. (For a detailed physical interpretation we refer to [7]). However, there are other useful ways in which the strength of a singularity can be tested. One such important criterion is to check whether it is a scalar polynomial singularity [10]. In the following, we show by means of an explicit example that when $\lambda = 0$, eventhough $R_{ab}K^aK^b$ does not diverge sufficiently fast, the Kretschmann scalar $K = R^{abcd}R_{abcd}$ can diverge along a non-spacelike trajectory meeting the singularity in past in the limit of approach to the singularity. Hence, a naked scalar polynomial singularity would result. It is not difficult to see that this represents certain general features of the situation when $M(u)$ is initially non-linear and we analyze the structure of the same in some detail.

We choose here a non-linear mass function which is representative of the class $M(u) \sim u^n$, > 1. The choice here is directed by the requirement that the equation of outgoing non-spacelike curve from the singularity should be simple, which helps towards an easier evaluation of the Kretschmann scalar K near the singularity. Consider $M(u)$ defined by,

$$2M(u) = \lambda u^{\alpha}\left(1 - 2\alpha\lambda u^{\alpha-1}\right) \tag{13}$$

where $\alpha > 1$ and $\lambda > 0$ is a constant. At $u = 0$, $M(u) = 0$ and we have flat space-time. The null radiation starts imploding at $u = 0$ till $u = T$ where T satisfies the condition,

$$T^{\alpha-1} < \frac{1}{2\lambda(2\alpha - 1)} \tag{14}$$

This ensures the positivity of $dM(u)/du$ and also that $M(u) > 0$. Thus the weak energy condition is satisfied. At $u = T$ we get the Schwarzschild configuration with mass $M_0 = M(T)$.

The radial null geodesics are again given by (5) which has a singular point at $u = 0$, $r = 0$. It is seen that for the mass function (13), an outgoing radial null geodesic meeting the singularity in past is given by,

$$r = \lambda u^{\alpha} \tag{15}$$

This integral curve meets the singularity with a tangent $r = 0$ and it is seen that the singularity is naked.

Analyzing the nature of this singularity in general, it is seen that one of the roots of the characteristic equation for (5) vanishes as a consequence of $(dM(u)\ /du)_{|u=0} = 0$. The structure of singularity therefore turns out to be more complicated than the case when $M(u)$ is linear and further information on the same can be obtained by writing (5) in the form (choosing $M(u) \sim u^\alpha, \alpha > 1$),

$$U^\alpha \frac{dR}{dU} = AR - \frac{U^{\alpha-1}}{2} + 0(R,U) \tag{16}$$

Here $U = u - 2r$, $UR = r$, A is a positive constant and $0(R,U)$ contains terms of order higher than one. It is seen that [9] the behavior of integral curves depends on the nature of α. When α is even, the singularity exhibits a col-node structure, i.e. given a neighbourhood of the singularity, it behaves like a col for integral curves in a certain region of (u,r) plane and like a node for rest of it. Hence, families of outgoing radial null geodesics can terminate at the singularity in past in such a case. On the other hand the singularity is a complete node when α is odd. In either case, there are families of integral curves that terminate at the singularity with either $r = 0$ or $u = 2r$ as tangent at the singularity.

Coming to the question of strength of the singularity, it is seen that the Kretschmann scalar diverges along all non-spacelike geodesics that meet the naked singularity with a definite tangent. For the case of the mass function given by (13), the behavior of K along the singular curve (15) is given by,

$$K = 48A\left(\frac{1-2\alpha\lambda u^{\alpha-1}}{\lambda^2}\right)^2\left(\log\frac{Bk}{2\lambda}\right)^{\frac{4\alpha}{\alpha-1}} \tag{17}$$

where A, B are positive constants. Clearly K diverges near the naked singularity as $k \to 0$. In a similar manner it can be shown using (3a,b) that for any $M(u) \sim u^\alpha$, $a > 1$, the scalar K diverges along all singular non-spacelike geodesics. It is thus seen that even though the singularity is not strong in the sense of condition (8), it is a strong curvature scalar polynomial singularity.

To conclude, we have shown that a general class of non-self-similar space-times, namely the Vaidya solutions with a non-linear mass term describing radiation collapse, contain a naked singularity which exhibits a strong curvature behavior. It follows that serious examples of naked singularities are not confined to self-similar spacetimes only as the case appeared from the work so far. Next, it is interesting to note that in *all* the examples available so far, a strong curvature naked singularity is associated with a node forming at the origin of the coordinates, allowing to conjecture that this will always be a nodal singularity. This conjecture is shown to be true here for the Vaidya class (1) for the range of λ given by $0 < \lambda \leq 1/8$. Conversely, a node need not always be a strong curvature singularity in the sense of (8) as shown by the example above, where the node does not satisfy (8) (which however is a scalar polynomial singularity).

I.H. Dwivedi acknowledges the support from C.S.I.R. and hospitality at T.I.F.R. while this work was done.

References

[1] R. Penrose, Riv. Nuovo Cimento Soc. Ital. Fis. **1**, 252 (1969)

[2] P. Yodzis, H.-J. Seifert and H. Muller zum Hagen, Commun. Math. Phys. **34**, 135 (1973), **37**, 29 (1974).

[3] B. Waugh and K. Lake, Phys. Rev. **D 38**, 1315 (1988), **40**, 2137 (1989); K. Rajagopal and K. Lake, Phys. Rev. **D35**, 1531 (1987).

[4] A. Ori and T. Piran, Phys. Rev. **D 42**, 1068 (1990).

[5] A. Papapetran, in 'A Random Walk in General Relativity' (ed. J. Krishna Rao et al) Wiley Eastern, New Delhi (1985); W. Israel, Can. J. Phys. **64**, 121 (1986).

[6] I.H. Dwivedi and P.S. Joshi, Class. Q. Grav. **6**, 1599 (1989).

[7] F.J. Tipler, C.J.S. Clarke and G.F.R. Ellis, in 'General Relativity and Gravitation' Vol. 2, ed. A. Held (New York, Penum), 1980.

[8] R.P.A.C. Newman, in 'Topological Properties and Global Structure of Spacetime' (ed. P.G. Bergmann and V. De Sabbata) Plenum, New York (1986).

[9] F.G. Tricomi, 'Differential Equations', Blackie & Sons, London (1961).

[10] S.W. Hawking and G.F.R. Ellis, 'The Large Scale Structure of Spacetime' Cambridge Univ. Press. Cambridge, 1973.