Preheating and Turbulence: Echoes of a Not So Quiet Universe

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We study the nonlinear decay of the inflaton which causes the reheating of the Universe in the transition from the inflationary phase to the radiation dominated phase, resulting in the creation of almost all matter constituting the present Universe. Our treatment allows us to follow the full dynamics of the system in a long time regime, and to describe not only the parametric resonance processes with nonlinear restructuring but also to characterize a final turbulent state in the dynamics by which the energy is nonlinearly transferred to all scales of the system with a consequent thermalization of the created matter.

Inflation has become a paradigm in Cosmology. So far all observational data collected from satellites and balloons have not imposed any considerable difficulty to the inflationary scenario. Although the physics underlying the beginning of inflation is still far from being understood, the *end* of inflation is a crucial issue that relates the transition from an almost empty and cold universe to a hot and radiation dominated universe.

The standard description of the end of inflation is known by reheating. Basically, it consists in the transfer of the energy stored in coherent oscillations of the inflaton to the production of particles; after interacting with each other, they come to a state of thermal equilibrium. However, several authors[1–4] have pointed out on the existence of a stage of parametric resonance in the beginning of the reheating – the preheating phase. For the sake of simplicity, let us consider a simple model of inflation in which the inflaton field has self interaction. At the end of inflation, the inflaton is composed by two pieces: a large and homogeneous component that performs coherent oscillations near the minimum of its effective potential, and quantum fluctuations developed during the inflation whose modes eventually become semiclassical. During the stage of coherent oscillations, energy is rapidly transferred from the homogeneous inflaton to some modes through the mechanism of parametric resonance. Physically, this means a huge production of particles and, due to the large growth of these modes, they soon cannot be considered perturbations as the nonlinearities come on the scene, and the linear approximation breaks down. We follow closely the dynamics of the decay of the inflaton in this nonlinear regime and our treatment alllows us to follow this process in a long time term. The nonlinearities are manifested by the backreation and rescattering of produced particles and mode-mode couplings and will be determinant for the achievement of the end of preheating signalized by a universe dominated by radiation in thermal equilibrium.

The aspect of paramount importance for a successful reheating is the nonlinear *transfer of energy* from the homogeneous inflaton field to its inhomogeneous modes. In this essay we shall focus on the connection between a necessary efficient energy transfer or *decay of energy* of the inflaton and the onset of turbulence.

In our specific problem of preheating, we consider the case of the inflaton with quartic potential $V(\phi) = \frac{1}{4}\lambda\phi^4$. The basic equation of our problem is the evolution of the inflaton field $\phi(\mathbf{x},t)$, in a spatially flat Friedmann-Robertson-Walker universe[1, 2]. Using the conformal time τ defined by $a(\tau)d\tau = \sqrt{\lambda}\phi_0(0)a(0)dt$, the conformal field $\varphi = \phi a(\tau)/\phi_0(0)a(0)$ and spatial coordinates $\mathbf{x} \to \sqrt{\lambda}\phi_0(0)a(0)\mathbf{x}$, it assumes the form

$$\varphi'' - \nabla^2\varphi - \frac{a''}{a}\varphi + \varphi^3 = 0 \qquad (1)$$

where a prime stands for the derivative with respect to τ and $\phi_0(0)$ is the homogeneous component of the inflaton field at $t = \tau = 0$. As we have mentioned previously, at the end of inflation the inflaton field undergoes the phase of coherent oscillations. It can be shown that the effective energy-momentum tensor of the inflaton in the theory $\frac{1}{4}\lambda\phi^4$ averaged over

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several oscillations is traceless[5], implying $a(\tau) \sim \tau$, and allowing us to set $a'' = 0$ in Eq. (1).

In this stage, the inflaton field has a large homogeneous component, $\varphi_0(\tau)$ together with its small fluctuations developed during the inflationary phase, so that

$$\varphi(\mathbf{x}, \tau) = \varphi_0(\tau) + \delta\varphi(\mathbf{x}, \tau). \quad (2)$$

At this point it will be interesting to make an analogy between this expression and the corresponding for the velocity field of a turbulent fluid flow,

$$u_i = U_i + \delta u_i, \quad (3)$$

where $U_i = \langle u_i \rangle$ is the mean velocity and δu_i is the turbulent fluctuating velocity. This last piece can be decomposed conveniently into Fourier modes, whose characteristic wave numbers that can be compared with the typical scale of the flow. As the turbulent flow evolves, it can be shown that energy is transferred from the mean flow – the homogeneous component of the velocity field – first to Fourier modes corresponding to large scales and then distributed for the next smaller scales[6, 7]. This process is a nonlinear redistribution of energy among the various scales of motion, and constitutes one of the key features of turbulence. Thus, by comparing Eqs. (2) and (3), $\phi_0(\tau)$ and $\delta\varphi(\mathbf{x}, \tau)$ play the role of U_i and δu_i, respectively. Also, during the preheating the energy stored in the homogeneous component of the inflaton is transferred to the several modes of the "turbulent" component $\delta\varphi(\mathbf{x}, \tau)$. As we are going to show, it will possible to distinguish the "large scale" and "small scales" modes present in the fluctuation $\delta\varphi(\mathbf{x}, \tau)$.

The integration of Eq. (1) will be performed in a two dimensional square box $\mathcal{D}$ of size L with periodic boundary conditions. For this task, we shall use the Galerkin method[6], which is largely applied in problems of turbulence. The basis functions $\{\psi_{\mathbf{k}}(\mathbf{x})\}$ that satisfies automatically the boundary conditions is suitably chosen as $\psi_{\mathbf{k}}(\mathbf{x}) = exp\left(\frac{2\pi i}{L}\,\mathbf{k}.\mathbf{x}\right)$, where $\mathbf{k} = (l, m)$ is the comoving momentum. The Galerkin-Fourier decomposition for the general inflaton field is

$$\begin{aligned} \varphi(\mathbf{x}, t) &= \sum_{l=-N}^{N} \sum_{m=-N}^{N} a_{lm}(\tau)\psi_{lm}(x, y) \\ &= \varphi_0(\tau) + \sum_{l,m} a_{lm}(\tau)\psi_{lm}(x, y), \end{aligned} \quad (4)$$

where N is the order of truncation to be chosen. The basis functions are orthogonal with respect to the inner product defined by $\langle \psi_{\mathbf{k}}, \psi_{\mathbf{l}} \rangle = \int_{\mathcal{D}} \psi_{\mathbf{k}} \psi_{\mathbf{l}}^* \, d^2\mathbf{x} = L^2\,\delta_{\mathbf{kl}}$. The modal coefficients a_{lm} are the classical analogue of amplitudes for processes of creation/anihilation of particles in QFT. Not all modal coefficients are independent, since by imposing the scalar field to be real, we arrive at $a_{lm}^* = a_{-l-m}$.

The remarkable advantage of the Galerkin method is to provide a dynamical system view of any physical system governed by partial differential equations. Usually, this reduction generates a low dimensional model that exhibits the same qualitative features of the exact system. In our case the Galerkin procedure is straightforward: insert the decomposition (4) into Eq. (1), the resulting equation being then projected into each $\mathbf{k}$th mode $\psi_{\mathbf{k}}(\mathbf{x})$. As a result, we obtain a set of equations for $a_{\mathbf{k}}(\tau)$ given by

$$a_{\mathbf{k}}''(\tau) + \omega_{\mathbf{k}}^2 a_{\mathbf{k}}(\tau) + \sum_{\mathbf{n},\mathbf{l}} a_{\mathbf{n}}(\tau) a_{\mathbf{l}}(\tau) a_{\mathbf{k}-\mathbf{n}-\mathbf{l}}(\tau) = 0, \quad (5)$$

where $\omega_{\mathbf{k}}^2 = \frac{4\pi^2}{L^2}\,\mathbf{k}^2$. A further decomposition of the modal coefficients into their real and imaginary parts is necessary, or $a_{\mathbf{k}}(\tau) = \alpha_{\mathbf{k}}(\tau) + i\,\beta_{\mathbf{k}}(\tau)$. The symmetry imposed on the modal coefficients produces $\alpha_{\mathbf{k}}(\tau) = \alpha_{-\mathbf{k}}(\tau)$ and, $\beta_{\mathbf{k}}(\tau) = -\beta_{-\mathbf{k}}(\tau)$ (note that the modal coefficient $\beta_{\mathbf{0}}(\tau)$ is zero, and $\alpha_{\mathbf{0}}(\tau) = \varphi_0(\tau)$ is the homogeneous component).

The corresponding general equations of motion for the homogeneous component $\alpha_{\mathbf{0}}(\tau)$, as well all other modes of the "turbulent" component $\delta\varphi(\mathbf{x}, \tau)$ are encompassed by Eq. (5), where the nonlinear terms have their origin in the mode-mode couplings including the homogeneous component. In the first stage of the preheating these nonlinear terms can be neglected, thus from Eq. (5) the usual description of this stage is recovered: the homogeneous component exhibits oscillatory behavior, whose exact solution is given in terms of an elliptic cosine with modulus $\sqrt{2}$, up to a rescale of the conformal time; the remaining modes satisfy Lamé equations and, depending on the value $\omega_{\mathbf{k}}$ assumes in the stability/instability chart for the Lamé equation, the modes undergo the regime of parametric resonance with exponential growth or are oscillatory. In plain terms, this means that the homogeneous component of the inflaton transfers considerable amount of energy to these resonant modes; this allows us to denote the resonant and the nonresonant modes as corresponding respectively to the large and small scales of the turbulent flow. Once the resonant modes ("large scale" modes) have grown considerably, the mode-mode couplings become relevant, and we may expect the beginning of the transfer ot energy to the initially nonresonant modes ("small scales" modes).

In order to go further with our analogy, it is now of utmost importance to integrate numerically Eq. (5), meaning the full evolution of the homogeneous component together with all other modes. The initial conditions are dictated by the physical conditions at the end of inflation as follows: due our rescaling

$\alpha_0(0) = 1$ and $\alpha_0'(0) = 0$; the initial conditions for the remaining modes are of quantum origin, more precisely, from the sub-Hubble modes at the end of inflation (see Ref. [3]). We have set $N = 2$ resulting in a dynamical system constituted by 25 independent second-order equations. Our guide to choose a suitable value for L is the linearized regime described after neglecting the nonlinear mode-mode couplings. Several modes were selected which undergo an initial phase of parametric resonance by considering the stability/instability chart for the Lamé equation that governs the evolution of the modes $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ in the linearized version[2]. Then, we set $L = 5\pi/\sqrt{2}$ such that all modes with $|l| = 2, |m| = 1$ are inside the instability band (in this case $\omega^2_{|2||1|} = 1.6$) and are amplified. We have performed numerical experiments[8] with sets of initial conditions determined by different values of λ ranging from 10^{-13} to 10^{-4} that determines[3] the initial amplitude of the modes $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$. The only observed physical feature due to the choice of distinct values of λ is the time required for the nonlinearities to become important.

In Fig. 1, the long time behavior of the homogeneous mode of the inflaton field, $\alpha_0(\tau)$, the resonant ("large scale") mode $\beta_{12}(\tau)$ and the nonresonant ("small scale") mode $\alpha_{11}(\tau)$ are depicted for $\lambda = 10^{-4}$. We have identified three distinct phases, where the particular duration of each phase depends on the value of λ. In the first phase that lasts from $\tau = 0$ to $\tau \approx 80$, $\alpha_0(\tau)$ oscillates with constant amplitude indicating that the mode-mode couplings have negligible influence. The behavior of the resonant mode $\beta_{12}(\tau)$ and the nonresonant mode $\alpha_{11}(\tau)$ are in agreement with the prediction provided by the linearized theory, i.e., the former experiences exponential growth while the later oscillates without changing considerably its amplitude.

In the second phase lasting from $\tau \approx 80$ to $\tau \approx 240$, the nonlinear mode-mode couplings start to alter the evolution of the homogeneous mode $\alpha_0(\tau)$, the resonant and nonresonant modes as well. Basically, this phase is characterized by the end of the parametric resonance with the beginning of the restructuring of the resonance. In other words, the distribution of energy from "large" to "smaller" scales enters into scene. As it can be seen from Fig. 1, $\alpha_0(\tau)$, the resonant mode $\beta_{12}(\tau)$; the nonresonant mode $\alpha_{11}(\tau)$ oscillates with increasing amplitude. Note that a minimum of the envelop of the oscillating mode $\alpha_0(\tau)$ coincides approximately with a maximum of the envelope of the resonant mode $\beta_{12}(\tau)$, and vice-versa, indicating a process of rescattering between these modes. Indeed, these nonlinear effects constitutes the first manifestations of what is known as the backreaction and rescattering. Then, we may

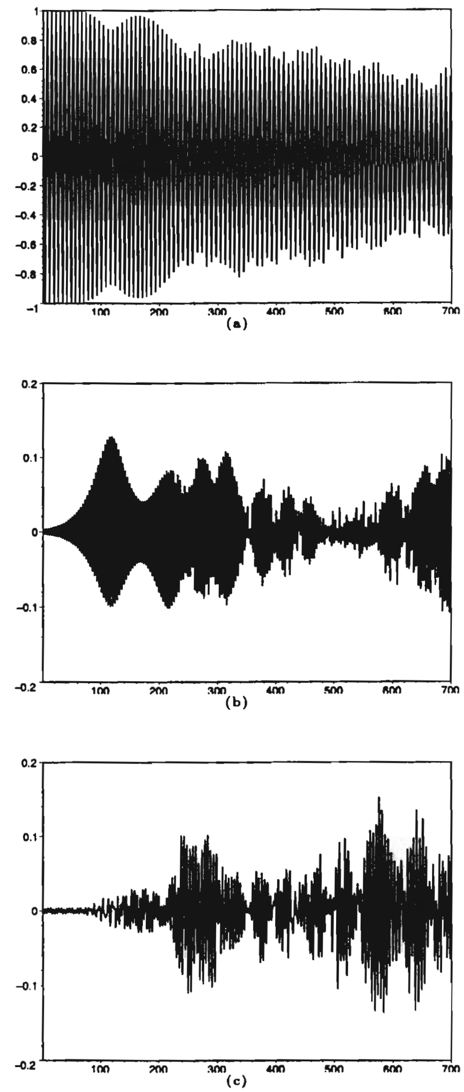

FIG. 1: The behavior of (a) homogeneous mode $\alpha_0(\tau)$, (b) a typical resonant mode $\beta_{12}(\tau)$, and (c) a nonresonant mode $\alpha_{11}(\tau)$ for $\lambda = 10^{-4}$. The overall dynamics is characterized by three phases: the linearized phase form $\tau = 0$ to $\tau \approx 80$, where the conventional preheating takes place; the *quasi-periodic* phase ($\tau \approx 80...240$), whose relevant feature is the end of the parametric resonance; and finally, the third phase - the *turbulent* phase. In this last phase there is no distinction between a resonant and a nonresonant mode due to the effective energy transfer from the inflaton to all mode.

denote this phase as the *quasi-periodic* phase.

The third phase initiates at $\tau \approx 240$ when the amplitude of the homogeneous mode reaches approximately a minimum of about 70% of its initial value. Remarkably, this feature was found for all values of λ in our numerical experiments. As it can be seen from Fig. 1, the homogeneous mode oscillations have an irregular pattern of modulated amplitude

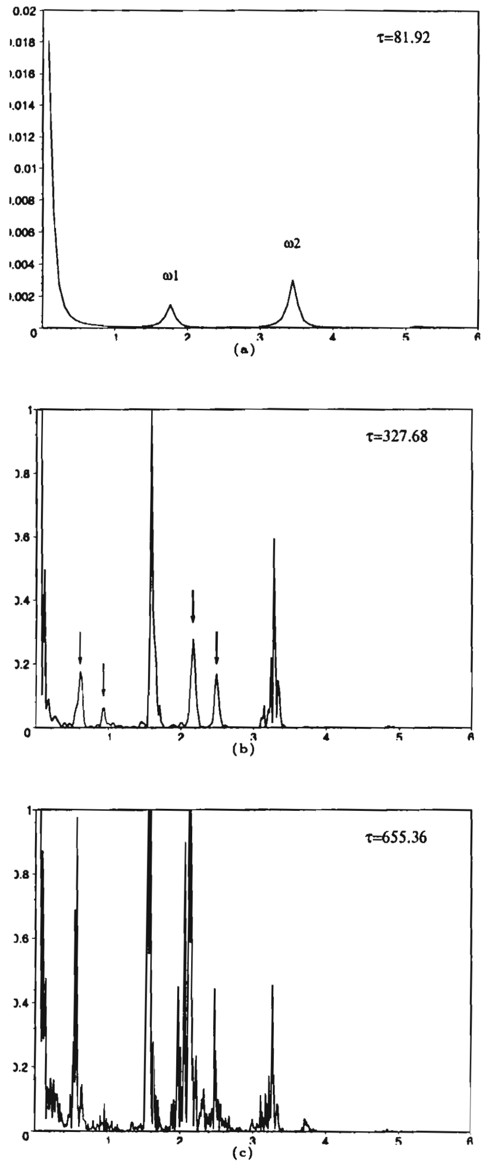

FIG. 2: Power spectra of the variance evaluated at $\tau = 81.92$, $\tau = 327.68$ and $\tau = 655.36$, the last two corresponding to the third phase. It is worth observing that this sequence show the period bifurcations $\frac{1}{3}\omega_1$, $\frac{1}{2}\omega_1$, for the first peak $\frac{5}{7}\omega_2$, and $\frac{5}{8}\omega_2$, for the second, as indicated by the arrows. This behavior is typical of the onset of turbulence found in fluid mechanics, as for instance in the Couette flow.

followed by a sequence of small bursts. Nonetheless, the most important aspect to be pointed out is the continuous decay of $\alpha_0(\tau)$. Concerning the resonant and nonresonant modes, it is no longer possible to make a distinction between them. These features are a dramatic consequence the action of nonlinearities, namely, the backreaction of the created particles into the homogeneous mode, as well as the rescattering of the produced particles into all other modes. Eventually, there will be no distinction whatsoever between the homogeneous mode and any other mode. Physically, this means that all modes will be in average equally populated, the particles dynamically transferring and distributing the energy among the modes producing, in this way, the thermalization. The thermalization process corresponds actually to the onset of a turbulent phase. A quantitative measure of the sum of all modal fluctuations produced about the homogeneous mode is given by the variance $\sigma^2 = \langle(\varphi - \alpha_0)^2\rangle = \sum(\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2)$, where α_0 is the expected value of the inflaton field. The power spectrum of the variance will be used to make a definite characterization of the final thermalization phase as the onset of a turbulent regime. Indeed, Fig. 2 depicts a sequence of the power spectra of the variance evaluated at several times, from the first to the third phase. This transition is constituted by period bifurcations, giving rise to approximate frequencies $\frac{1}{3}\omega_1$, $\frac{1}{2}\omega_1$, $\frac{5}{7}\omega_2$ and $\frac{5}{8}\omega_2$, with $\omega_1 \simeq 1.77$ and $\omega_2 \simeq 3.45$, characteristic of a typical road to turbulence[7]. From the power spectrum for $\tau = 655.36$, it can be noted the presence of broad band portions, despite the presence of sharp frequencies, which tend to disappear asymptotically. This last phase is denoted as the ***turbulent*** phase.

In conclusion, the Galerkin projection method establishes a clear dynamical picture of the nonlinear decay of the inflaton with potential $V(\phi) = \frac{1}{4}\lambda\phi^4$ as the dynamics of a countable set of nonlinear coupled harmonic oscillators. The process develops in three distinct phases[9], starting from the linear regime of parametric resonance to a final thermalization process. An essential feature of the process is the transition from the quasi-periodic phase, in which the parametric resonance is suppressed, towards a turbulent regime characterized by a highly effective transfer of energy from the homogeneous mode to "large" scale modes and then to "small" scale modes, due to the nonlinear coupling of the modes that dominates the dynamics in a long-time term. As we have mentioned, this last aspect is a remarkable property observed in turbulent fluid dynamics in which an effective energy transfer from the mean flow to the turbulent flow takes place[6]. Therefore, as a consequence, all modes eventually become statistically equally populated indicating the state of thermalization.

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[9] It is important to remark that the presence of these three phases is robust with respect to the change of L, only the time scales for each phase are modified.